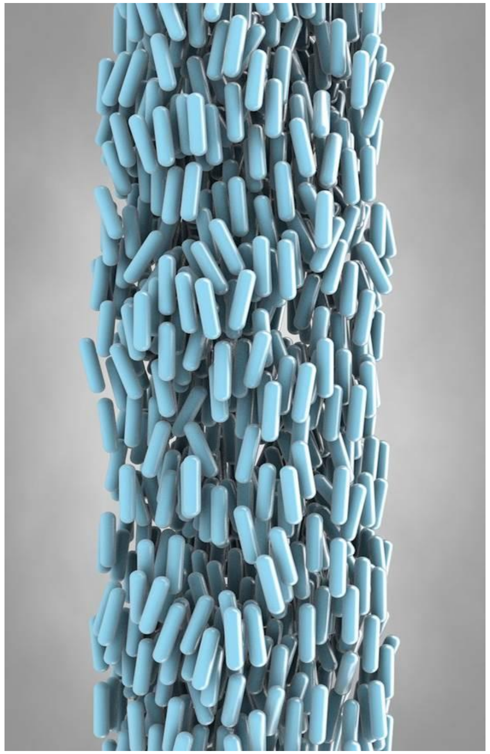
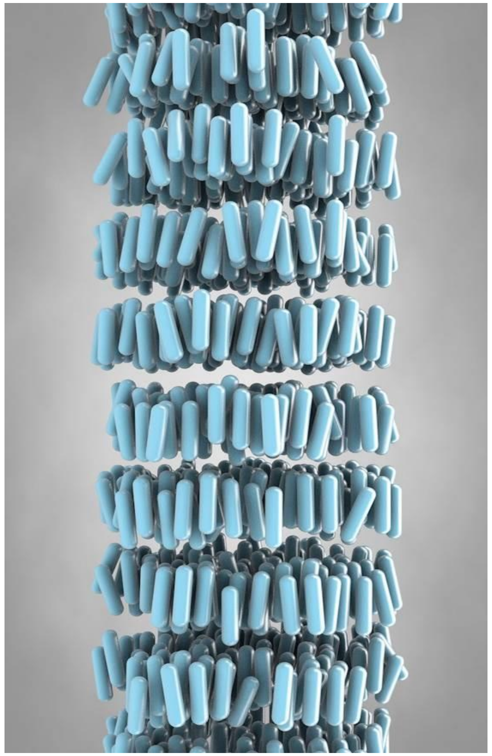
Overview of lecture 1

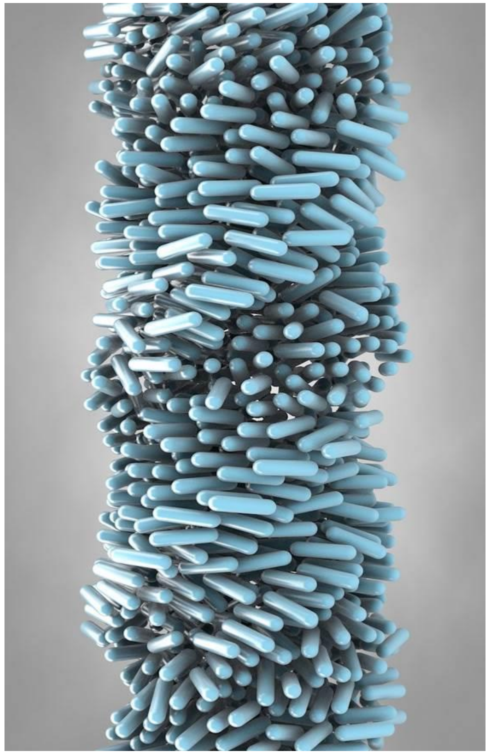
- The self-organization of rod-like molecules gives rise to hybrid states or mesophases, that are half-liquid (they can flow) and half-solid (they have anisotropic properties) = liquid crystals.



Nematic



Smectic A



Cholesteric

Overview of lecture 1

- The simplest mesophase is the nematic phase, in which there is no positional order and a long-range orientational order: the average local orientation of the molecular axis is given by the director field ***n***.

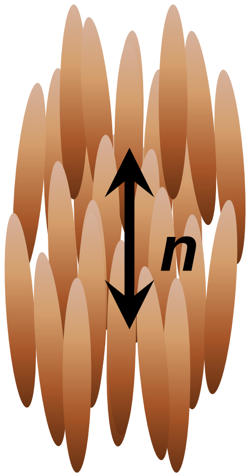
- The degree of orientational order is described by the scalar order parameter S

$$S = < P_2(\cos\theta) > = \left\langle \frac{3}{2}\cos^2\theta - \frac{1}{2} \right\rangle$$

- Anisotropy $\Rightarrow$ Physical properties are different along ***n*** and orthogonally to it.

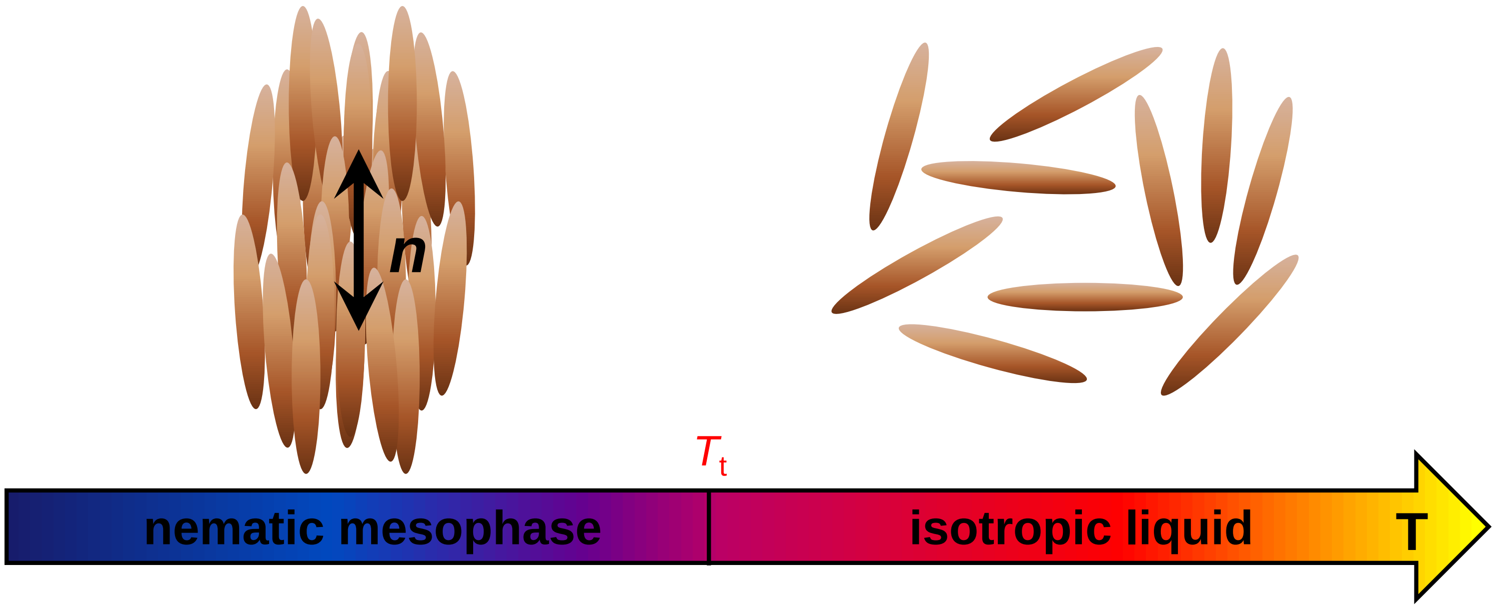
Example: optics

$$\boldsymbol{D} = \overline{\overline{\varepsilon}}\boldsymbol{E} \qquad \overline{\overline{\varepsilon}} = \begin{pmatrix} \varepsilon_\perp & 0 & 0 \\ 0 & \varepsilon_\perp & 0 \\ 0 & 0 & \varepsilon_{||} \end{pmatrix}$$



Overview of lecture 1

- The isotropic-nematic phase transition presents a spontaneous symmetry breaking:

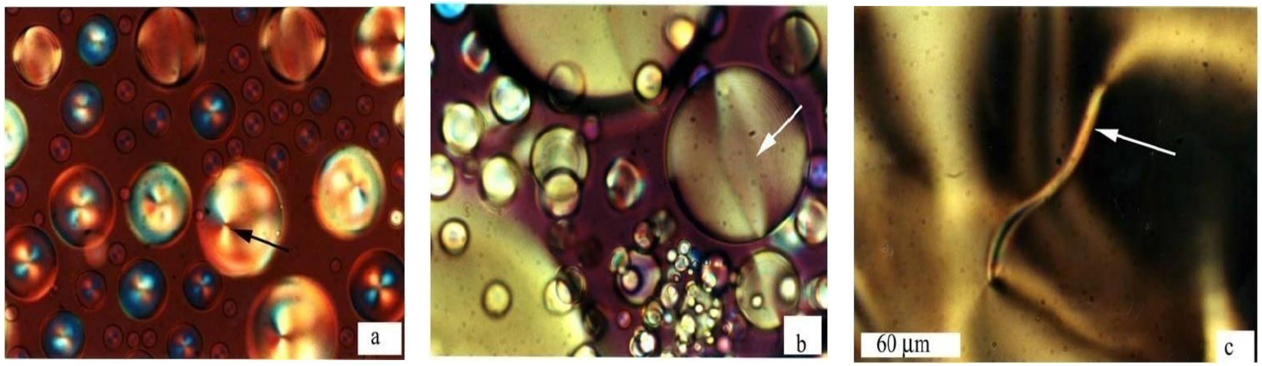


	nematic mesophase		isotropic liquid
	no positional order	↔	*no positional order*
	orientational order	**SSB**	***no orientational order***
	$0 \leq S \leq 1$		$S = 0$
symmetry group	***O(2)* x *T(3)***		***SO(3)* x *T(3)***

Overview of lecture 1

M Kleman et al. Phil Mag 86, 4117 (2006)

- The isotropic-nematic phase transition presents a spontaneous symmetry breaking leading to defect production:



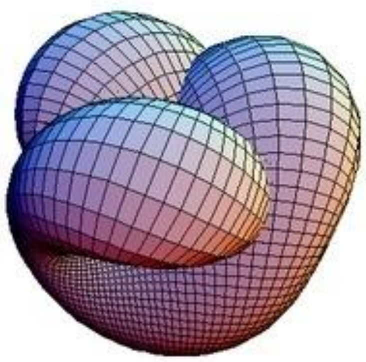
isotropic liquid

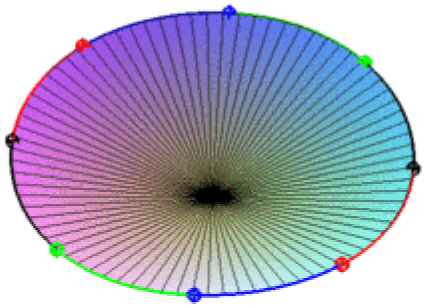
Domino cascade = SSB

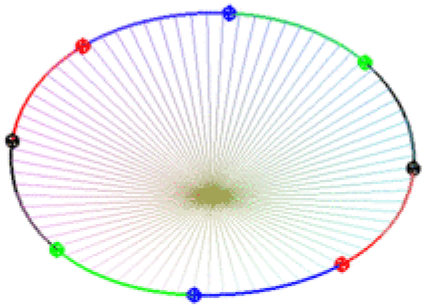
nematic mesophase
with a defect

Overview of lecture 1

- The order parameter space is the real projective plane $M = SO(3)/O(2) \sim S^2/Z_2$ ↔ « Boy surface »







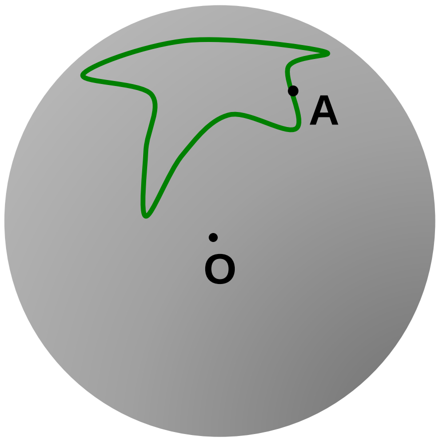
- The content of homotopy groups predicts the kind of defects that may appear in the mesophase:

$\pi_0(M)=I$	$\pi_1(M)=Z_2$	$\pi_2(M)=\mathbb{Z}$	$\pi_3(M)=\mathbb{Z}$
No domain wall	*Line defects: **disclinations**…*	*Monopoles: hedgehogs…*	*Textures: skyrmions…*

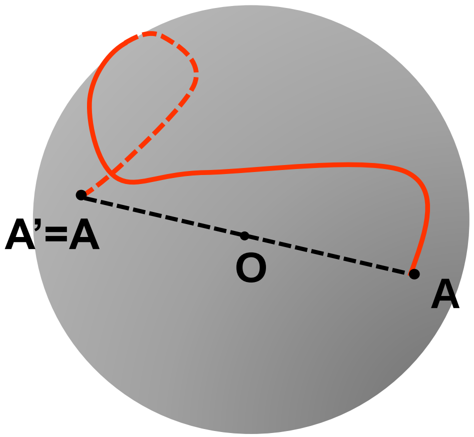
Overview of lecture 1

- The content of the first homotopy group means that there are only two equivalence classes for the linear defects:

unstable



$\pi_1(\mathbb{R}P^2)=Z_2=\{0,1\}$

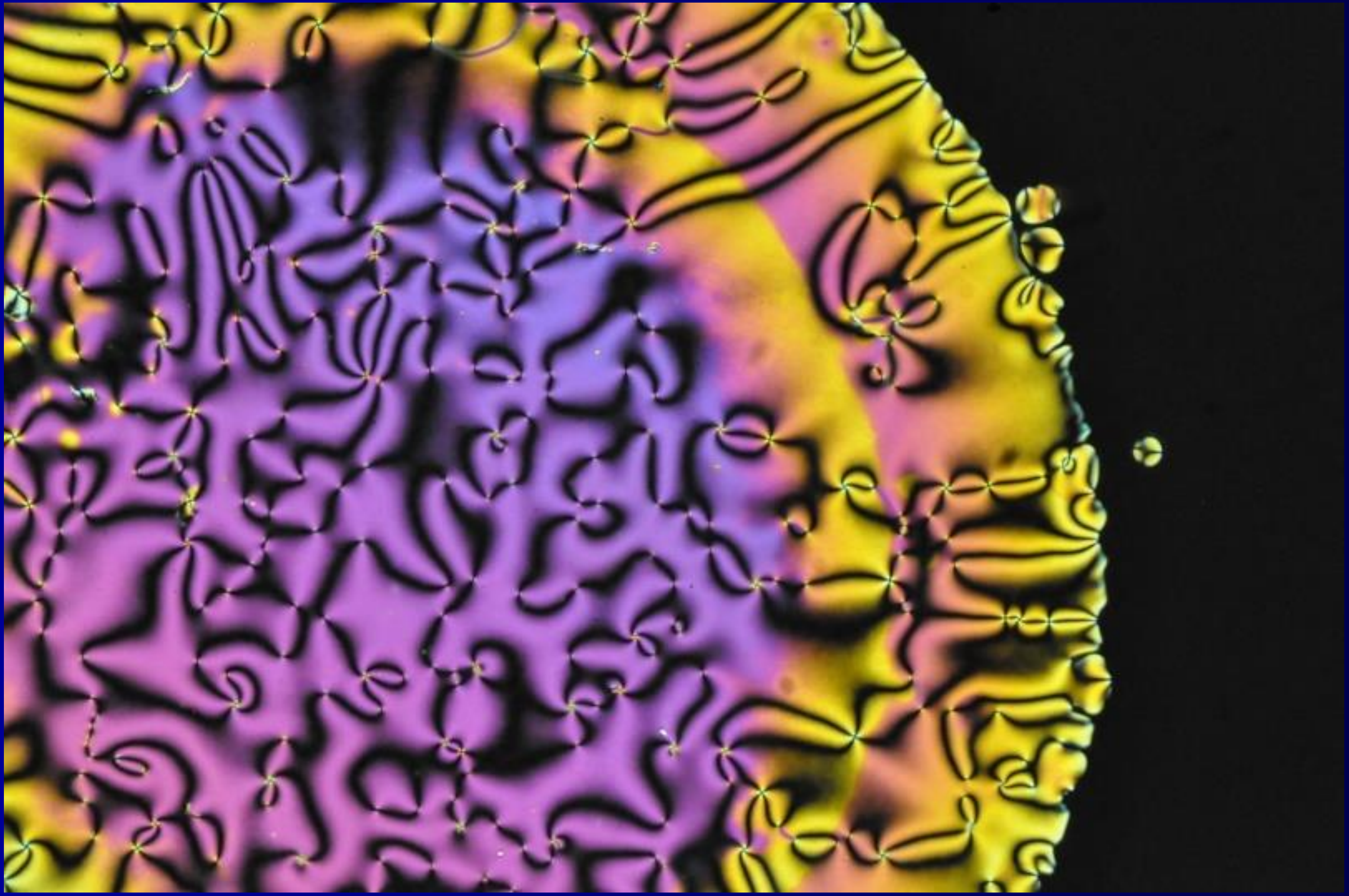


stable

These two classes of defects can be combined according to the algebra of $\mathbb{Z}/2\mathbb{Z}$:

0+0=0 **0+1=1** **1+1=0**

Part II. Topological defects in nematics



Nematoelasticity in a nutshell

- Consider a given director field $\boldsymbol{n}_0(\boldsymbol{r})$. Deformations about that configuration are orthogonal to $\boldsymbol{n}_0(\boldsymbol{r})$ as

$$\boldsymbol{n}_0.\boldsymbol{n}_0 = 1 \Rightarrow \boldsymbol{n}_0.\delta\boldsymbol{n} = 0$$

To simplify, if one takes $\boldsymbol{n}_0(\boldsymbol{r})=\boldsymbol{e}_3$, then $\delta\boldsymbol{n}=(\delta n_1, \delta n_2, 0)$. Let be $\boldsymbol{n}=\boldsymbol{n}_0+\delta\boldsymbol{n}$, a Taylor expansion gives:

$$\begin{pmatrix} \delta n_1 \\ \delta n_2 \end{pmatrix} = \begin{pmatrix} \dfrac{\partial n_1}{\partial x_1}\delta x_1 + \dfrac{\partial n_1}{\partial x_2}\delta x_2 + \dfrac{\partial n_1}{\partial x_3}\delta x_3 + \ldots \\ \dfrac{\partial n_2}{\partial x_1}\delta x_1 + \dfrac{\partial n_2}{\partial x_2}\delta x_2 + \dfrac{\partial n_2}{\partial x_3}\delta x_3 + \ldots \end{pmatrix}$$

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$$\begin{pmatrix}\delta n_1\\ \delta n_2\end{pmatrix} = \frac{1}{2}\begin{bmatrix}\dfrac{\partial n_1}{\partial x_1}+\dfrac{\partial n_2}{\partial x_2} & 0\\ 0 & \dfrac{\partial n_1}{\partial x_1}+\dfrac{\partial n_2}{\partial x_2}\end{bmatrix}\begin{pmatrix}\delta x_1\\ \delta x_2\end{pmatrix} + \frac{1}{2}\begin{bmatrix}0 & \dfrac{\partial n_1}{\partial x_2}-\dfrac{\partial n_2}{\partial x_1}\\ -\dfrac{\partial n_1}{\partial x_2}+\dfrac{\partial n_2}{\partial x_1} & 0\end{bmatrix}\begin{pmatrix}\delta x_1\\ \delta x_2\end{pmatrix}$$

$$+\frac{1}{2}\begin{bmatrix}\dfrac{\partial n_1}{\partial x_1}-\dfrac{\partial n_2}{\partial x_2} & \dfrac{\partial n_1}{\partial x_2}+\dfrac{\partial n_2}{\partial x_1}\\ \dfrac{\partial n_1}{\partial x_2}+\dfrac{\partial n_2}{\partial x_1} & -\dfrac{\partial n_1}{\partial x_1}+\dfrac{\partial n_2}{\partial x_2}\end{bmatrix}\begin{pmatrix}\delta x_1\\ \delta x_2\end{pmatrix} + \delta x_3\begin{pmatrix}\dfrac{\partial n_1}{\partial x_3}\\ \dfrac{\partial n_2}{\partial x_3}\end{pmatrix}$$

Nematoelasticity in a nutshell

- First term:

$$\frac{1}{2}\begin{bmatrix} \frac{\partial n_1}{\partial x_1}+\frac{\partial n_2}{\partial x_2} & 0 \\ 0 & \frac{\partial n_1}{\partial x_1}+\frac{\partial n_2}{\partial x_2} \end{bmatrix} \rightarrow f_1 = \frac{1}{2}\left(\mathrm{div}\,\boldsymbol{n}\right)$$

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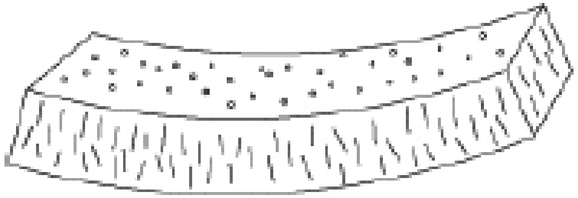
⟺

« splay »



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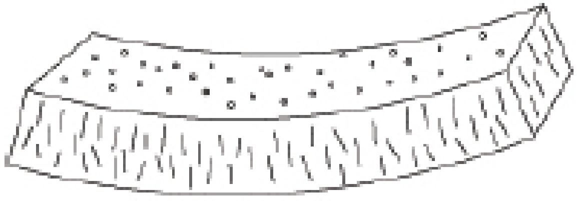
$\Longleftrightarrow$

« splay »



- Second term:

$$\frac{1}{2}\begin{bmatrix} 0 & \frac{\partial n_1}{\partial x_2}-\frac{\partial n_2}{\partial x_1} \\ -\frac{\partial n_1}{\partial x_2}+\frac{\partial n_2}{\partial x_1} & 0 \end{bmatrix} \rightarrow f_2 \simeq \frac{1}{2}\left(\boldsymbol{n}.\mathrm{curl}\,\boldsymbol{n}\right)$$

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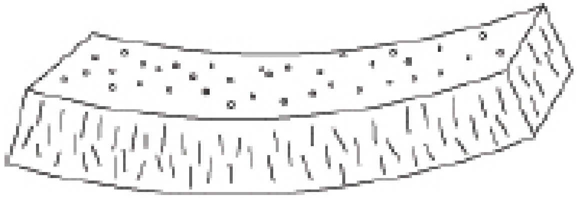

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« twist »

Nematoelasticity in a nutshell

- Third term:

« bend »

$$\frac{1}{2}\begin{bmatrix} \frac{\partial n_1}{\partial x_1} - \frac{\partial n_2}{\partial x_2} & \frac{\partial n_1}{\partial x_2} + \frac{\partial n_2}{\partial x_1} \\ \frac{\partial n_1}{\partial x_2} + \frac{\partial n_2}{\partial x_1} & -\frac{\partial n_1}{\partial x_1} + \frac{\partial n_2}{\partial x_2} \end{bmatrix} \rightarrow f_3 \simeq \frac{1}{2}\left(\boldsymbol{n} \wedge \mathbf{curl}\, \boldsymbol{n}\right) \Longleftrightarrow$$

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« bend »

- Similarwise to the harmonic oscillator, the (simplest) Frank-Oseen free energy describing nematoelasticity writes as

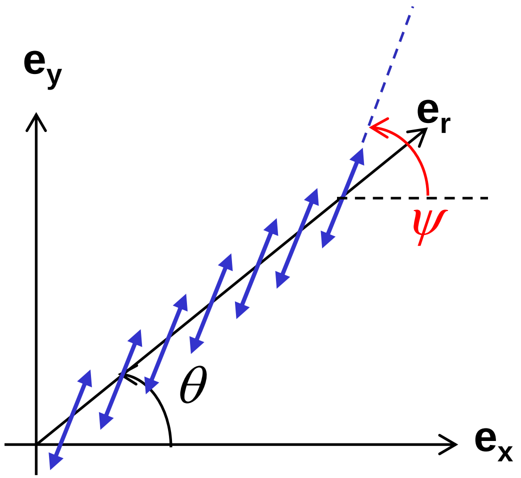
$$F=\frac{1}{2}KX^2 \simeq \frac{1}{2}K_1\left(\mathrm{div}\,\boldsymbol{n}\right)^2+\frac{1}{2}K_2\left(\boldsymbol{n}.\mathbf{curl}\,\boldsymbol{n}\right)^2+\frac{1}{2}K_3\left(\boldsymbol{n}\wedge\mathbf{curl}\,\boldsymbol{n}\right)^2$$

Planar distorsions

- "One constant approximation" (isotropic elasticity) : $K_1 \sim K_2 \sim K_3 = K \sim \frac{E_0}{L}$

$$\Rightarrow F = \frac{K}{2}\left[(\mathrm{div}\,\boldsymbol{n})^2 + (\boldsymbol{n}.\mathbf{curl}\,\boldsymbol{n})^2 + (\boldsymbol{n}\wedge\mathbf{curl}\,\boldsymbol{n})^2\right] = \frac{K}{2}\left[(\mathrm{div}\,\boldsymbol{n})^2 + (\mathbf{curl}\,\boldsymbol{n})^2\right]$$

- For planar configurations of the director field (x-y), tedious calculations give



$$\boldsymbol{n}(r,\theta) = \begin{pmatrix} \cos\psi(r,\theta) \\ \sin\psi(r,\theta) \\ 0 \end{pmatrix} \quad \Rightarrow F = \frac{K}{2}(\mathbf{grad}\,\psi)^2$$

Euler-Lagrange equation for the Frank-Oseen energy then reduces to

$$\Delta\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = 0$$

Planar distorsions

- One seeks simple solutions that are not depending on r : $\frac{d^2\psi}{d\theta^2} = 0 \Rightarrow \boxed{\psi(\theta) = m\theta + \psi_0}$

$\Rightarrow$ After a full turn about the z-axis, $\oint_{\theta=2\pi} d\psi = 2\pi m$

m = winding number (in $\mathbb{R}$)

$2m$ = Frank index

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- Constraints on *k* : the direction of ***n*** is well-defined at each point $\Rightarrow \oint_{\theta=2\pi} d\psi = 2\pi m = k\pi \quad k \in \mathbb{Z}$

(Z_2 symmetry of the nematic state)

$$\Rightarrow m = \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 2 \ldots$$

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Whole-numbered disclinations

- **What does a distorted nematic locally look like ?**

It depends on the parameters (m, ψ_0). Let us learn a little more from several examples:

$$m = +1, \psi_0 = 0 \Rightarrow \boldsymbol{n}(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} = \mathbf{e}_r$$

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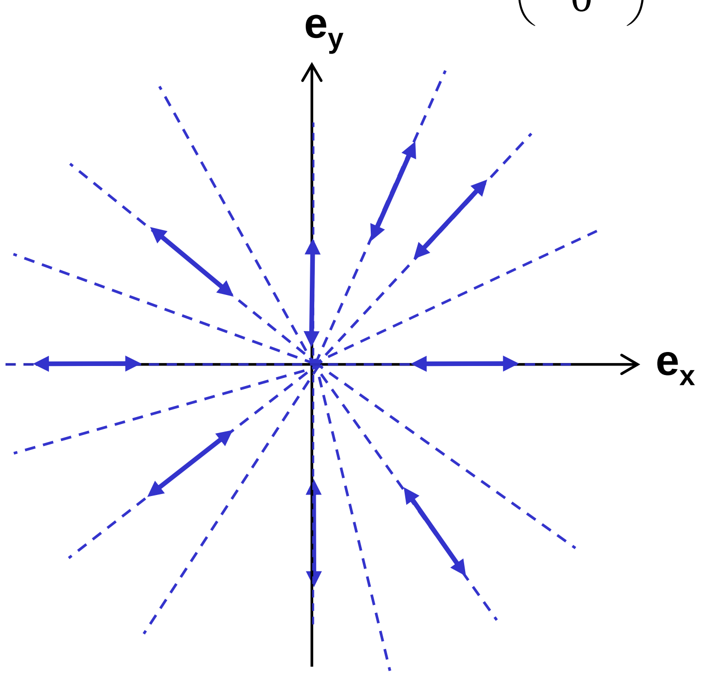

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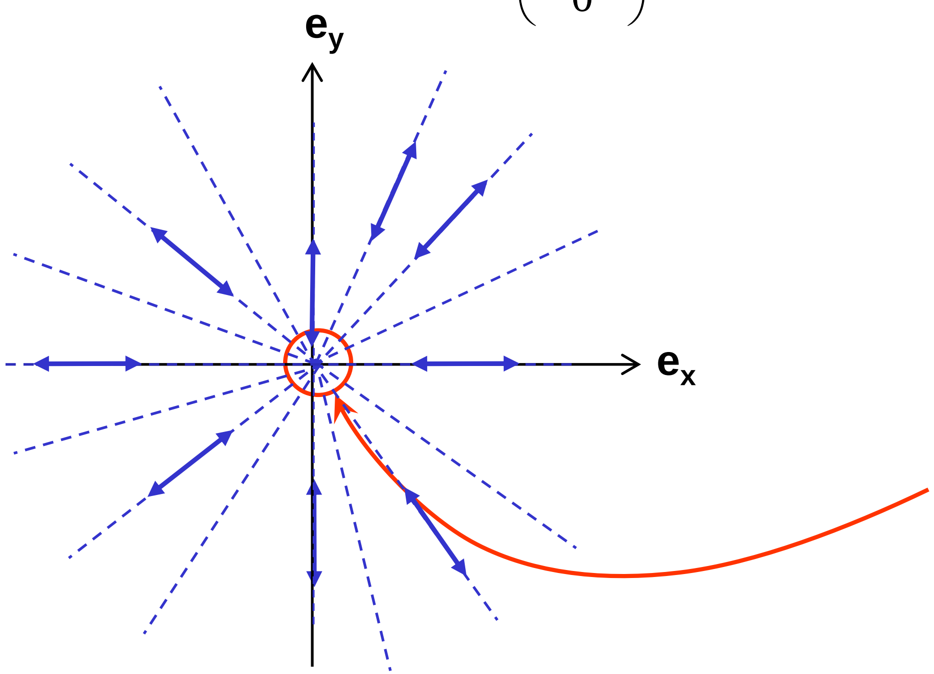

singularity of orientation along this line
= linear « defect »
= « disclination line »

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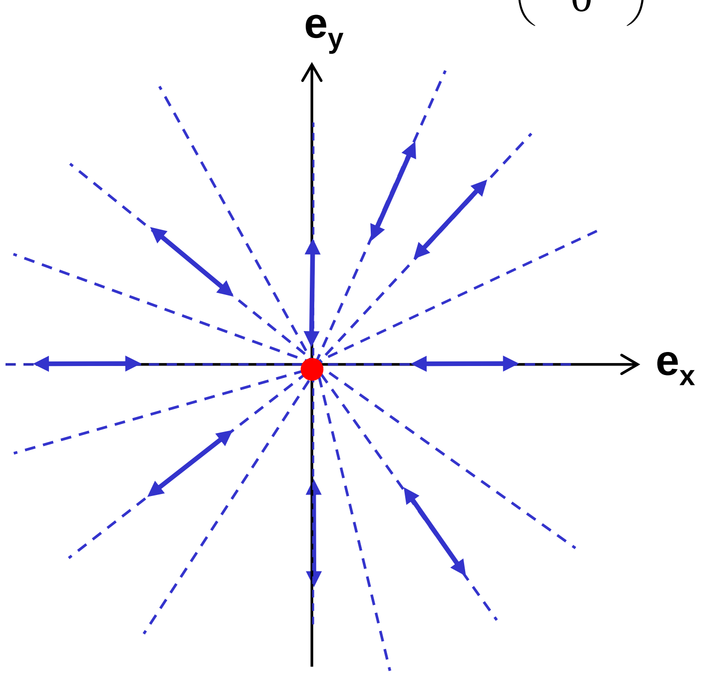

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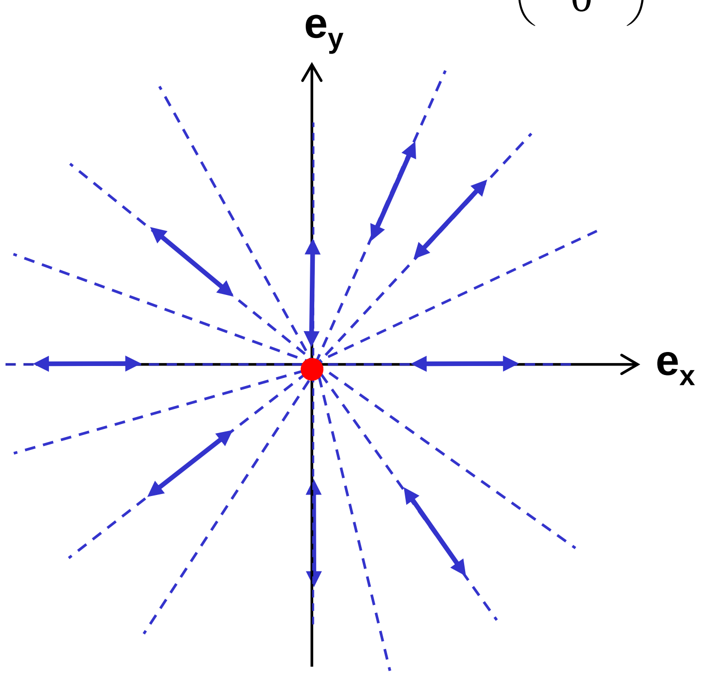

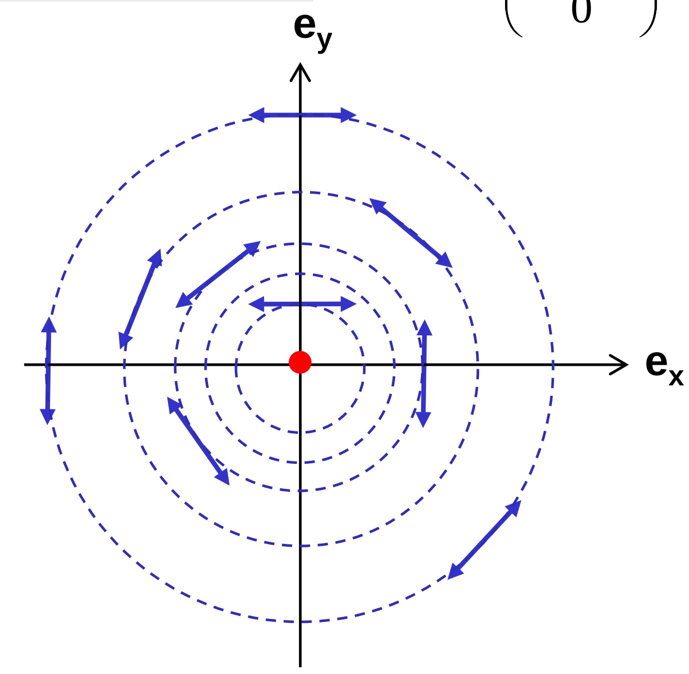

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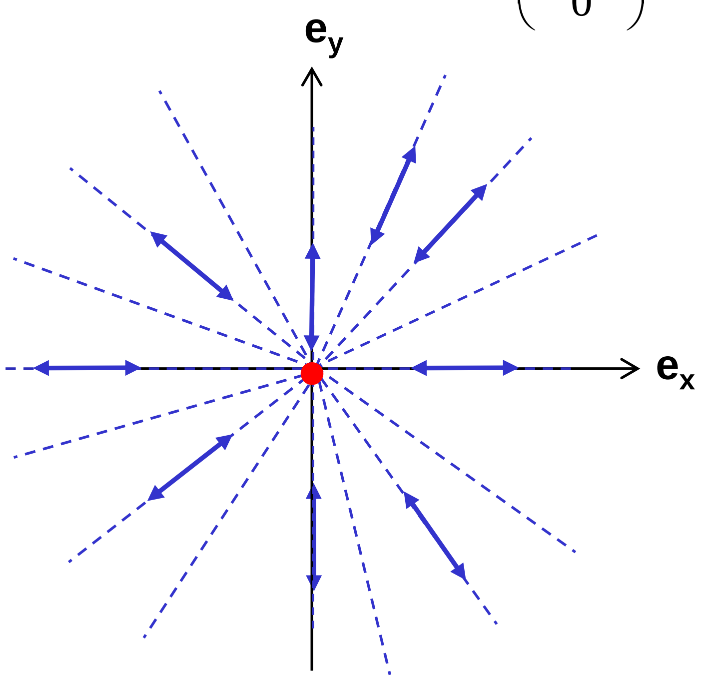

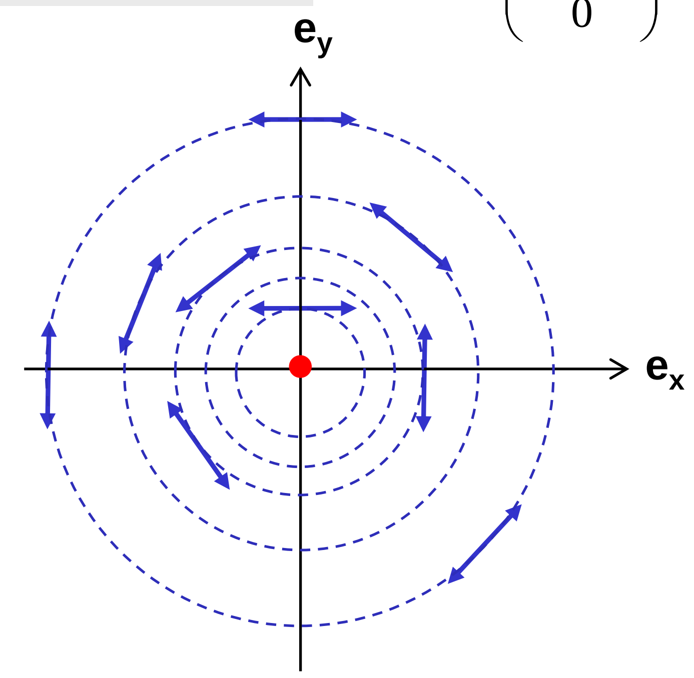

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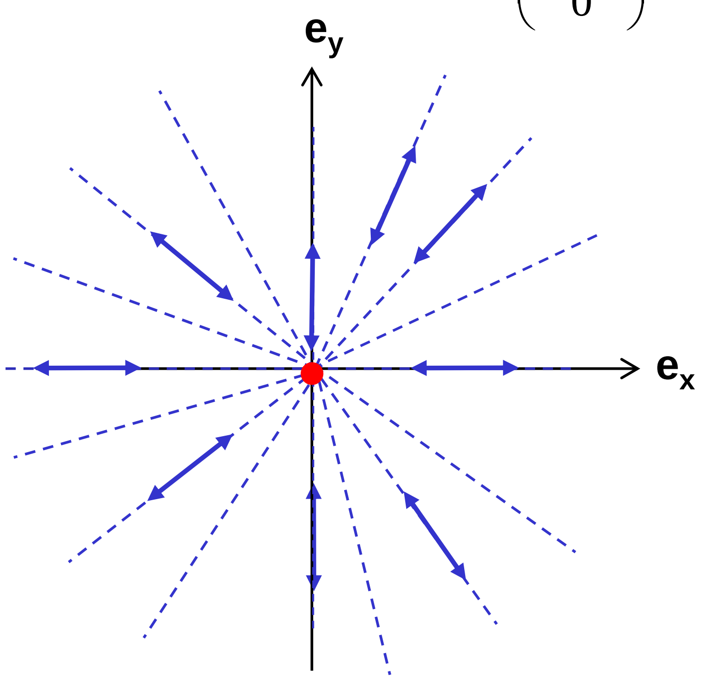
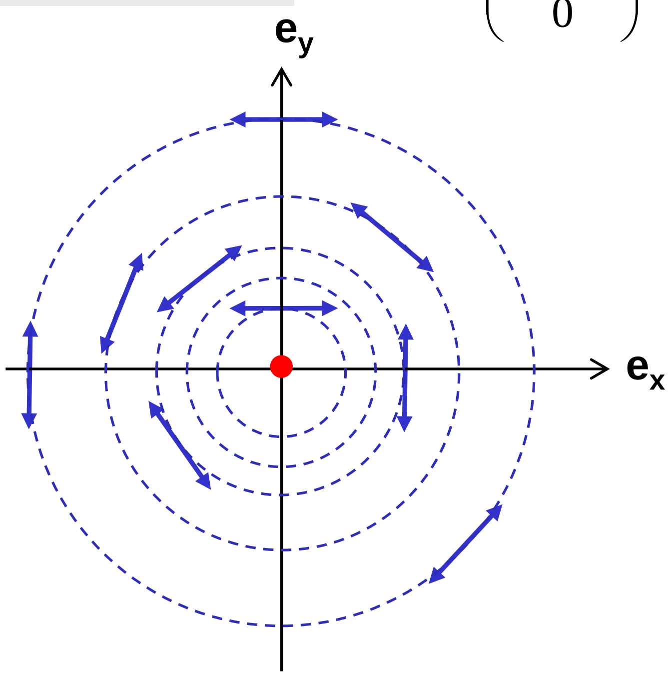


Topologically equivalent (D≥2)



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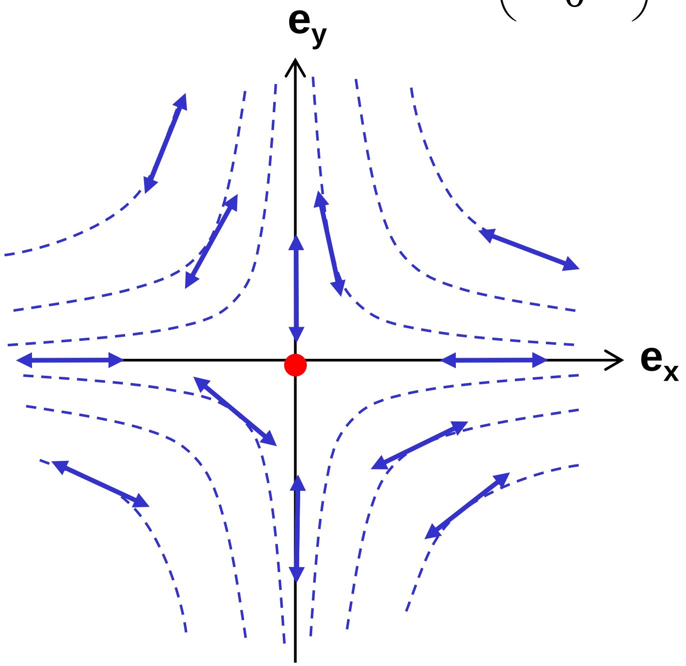

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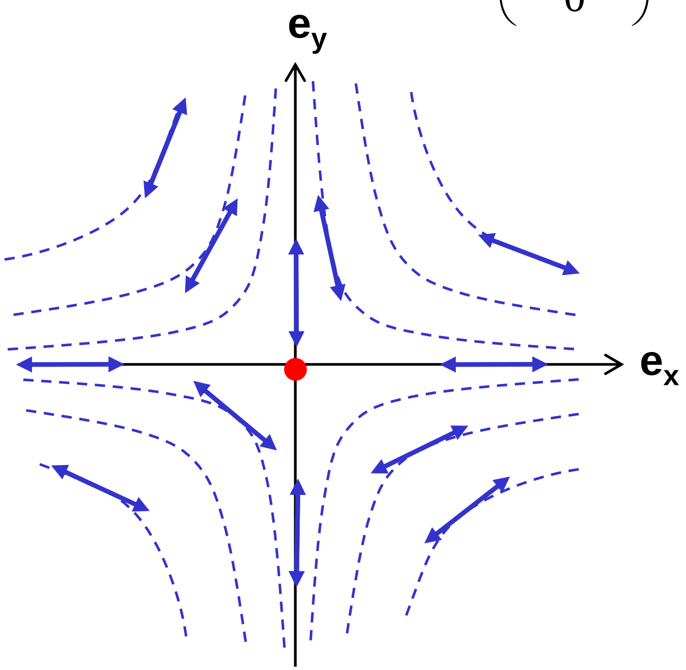

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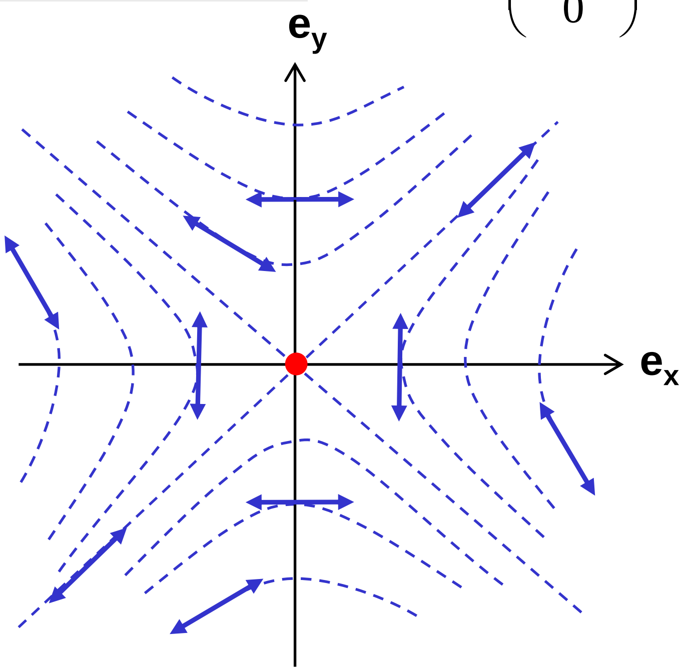

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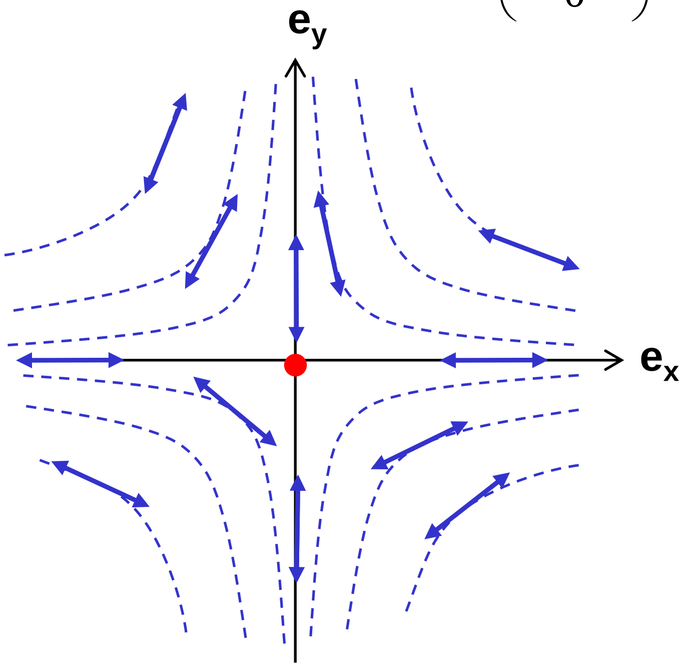
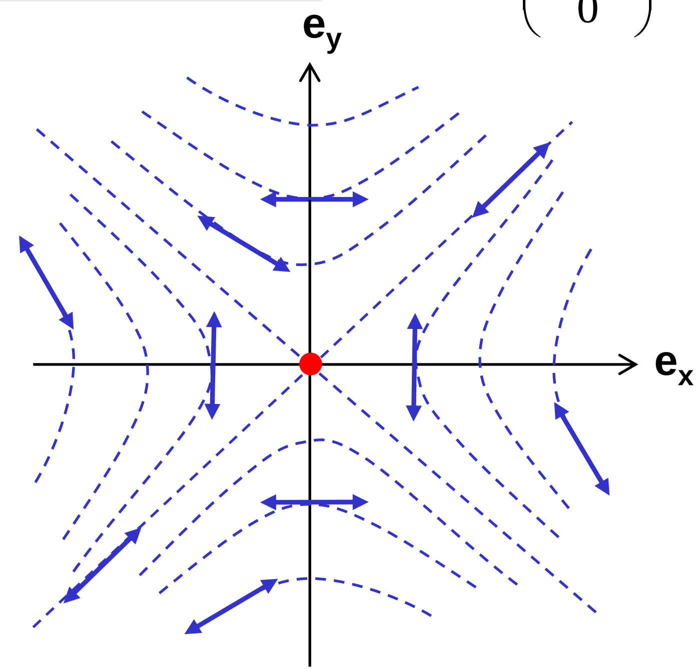


$$m = -1, \psi_0 = \pi / 2 \Rightarrow \boldsymbol{n}(\theta) = \begin{pmatrix} \sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$$



Topologically equivalent (D≥2)

Whole-numbered disclinations

- **Is m the label of π_1 equivalence classes ?**

Whole-numbered disclinations

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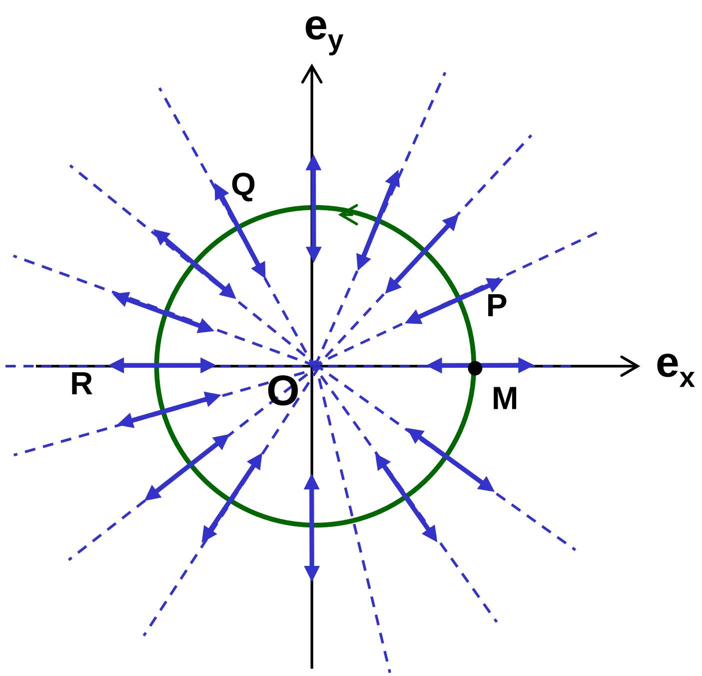
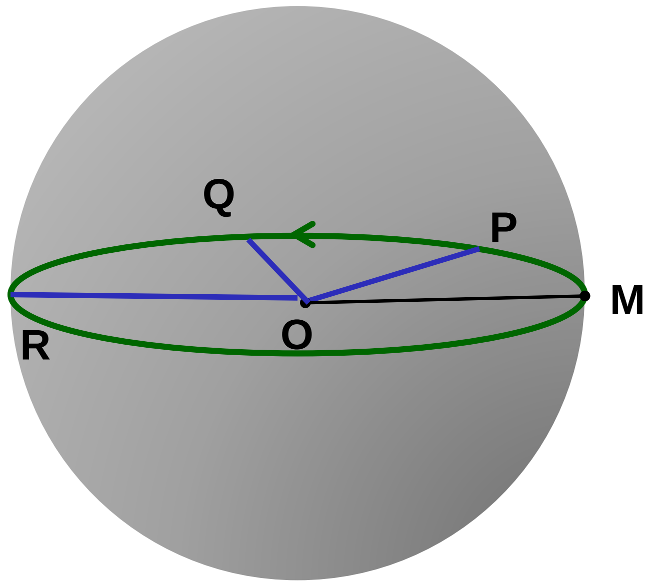


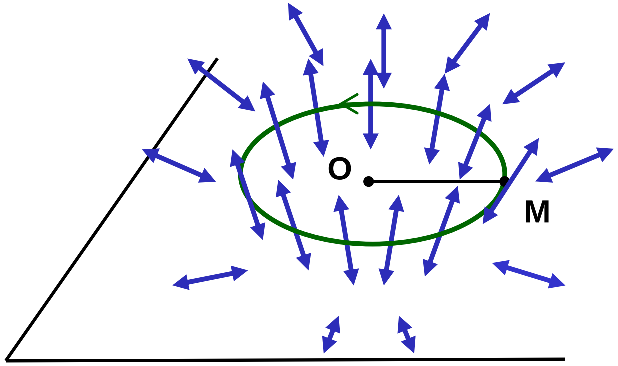


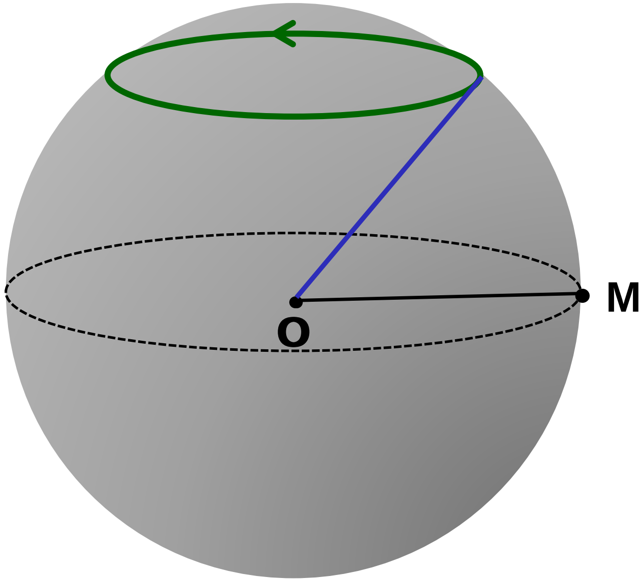
$\pi_1(\mathbb{R}P^2)=Z_2=\{0,1\}$

Whole-numbered disclinations

- **Is m the label of π_1 equivalence classes ?**

The « magic trick » = escape in the third dimension !





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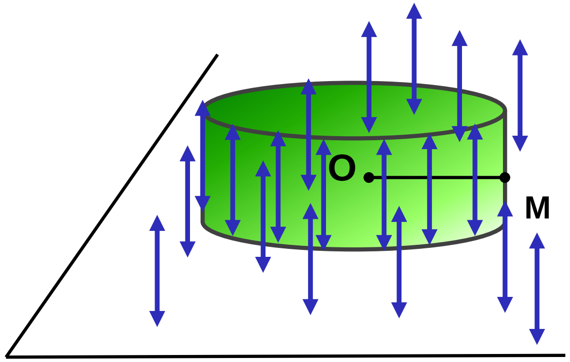
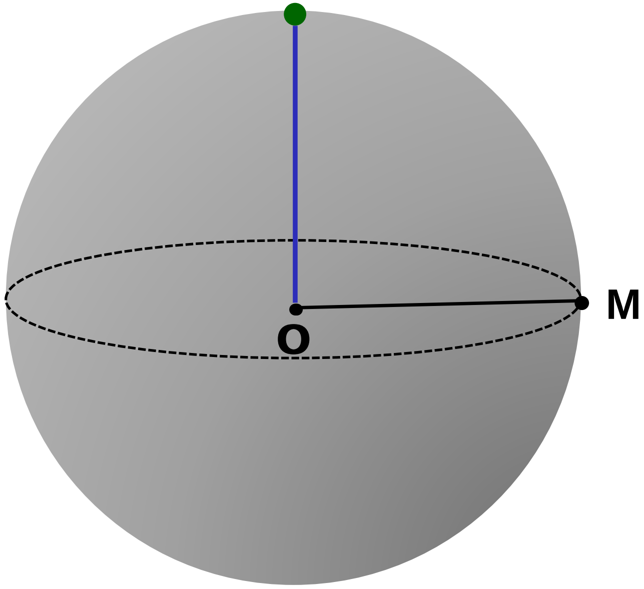


Regular ribbon



$\pi_1(\mathbb{R}P^2)=Z_2=\{0,1\}$

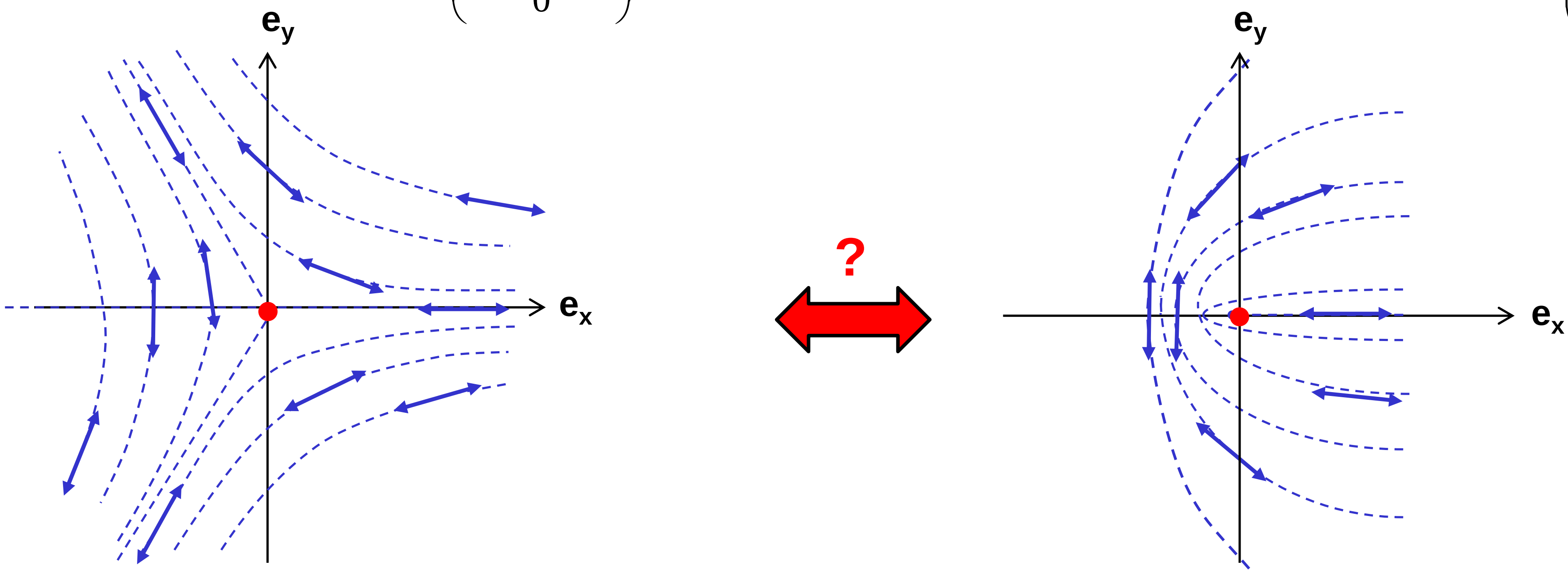
The homotopy loop can be shrunk to a point = **topologically removable defect**

Likewise for m=-1 $\Rightarrow$ Disclinations of integer strengths belong to the trivial homotopy class N= **0**.

Moebius disclinations

$$m=-1/2, \psi_0=0 \Rightarrow \boldsymbol{n}(\theta)=\begin{pmatrix}\cos(\theta/2)\\ -\sin(\theta/2)\\ 0\end{pmatrix}$$

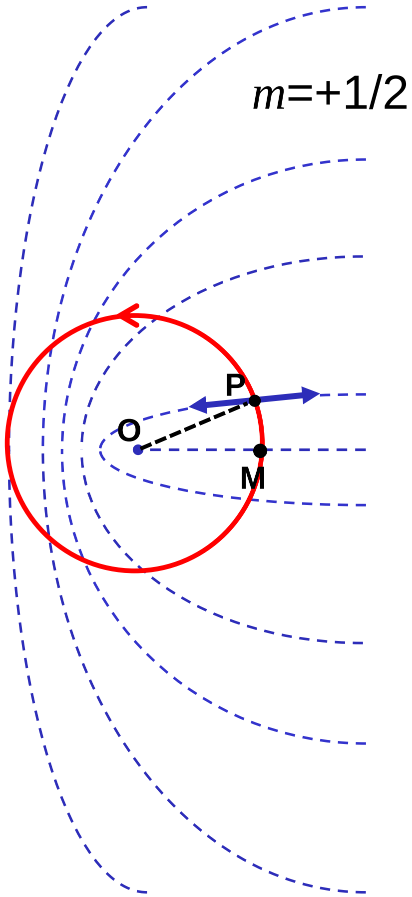
$$m=+1/2, \psi_0=0 \Rightarrow \boldsymbol{n}(\theta)=\begin{pmatrix}\cos(\theta/2)\\ \sin(\theta/2)\\ 0\end{pmatrix}$$



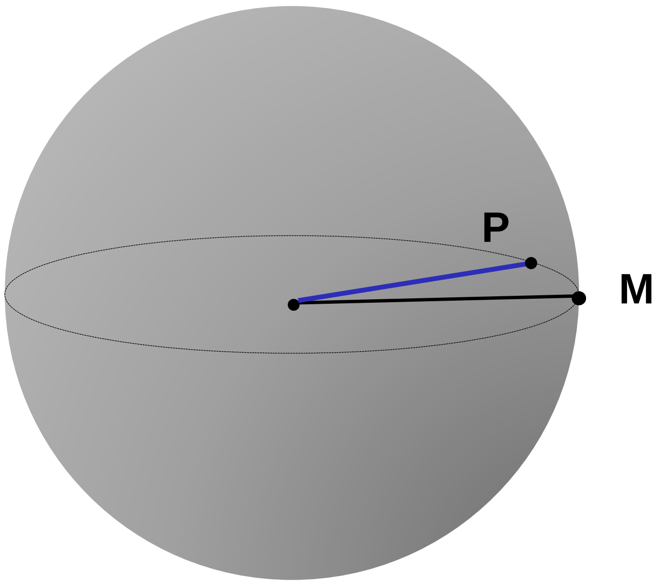
« *Trefoil* »

« *Comet* »

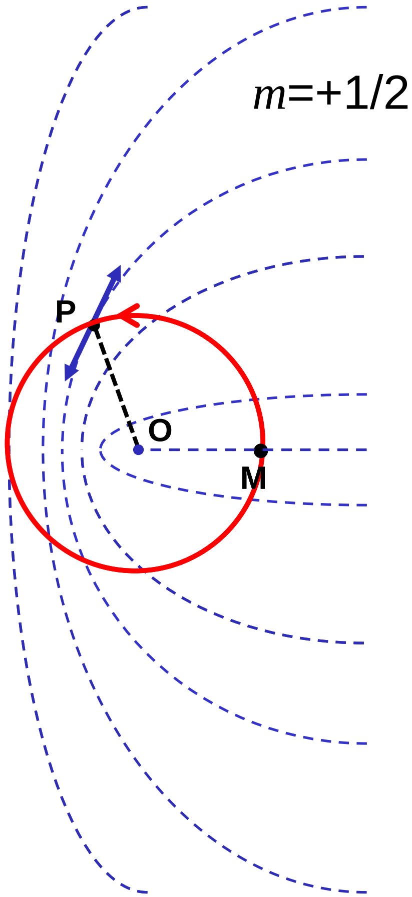
Moebius disclinations



$$\pi_1(\mathbb{R}P^2)=Z_2=\{0,1\}$$



Moebius disclinations



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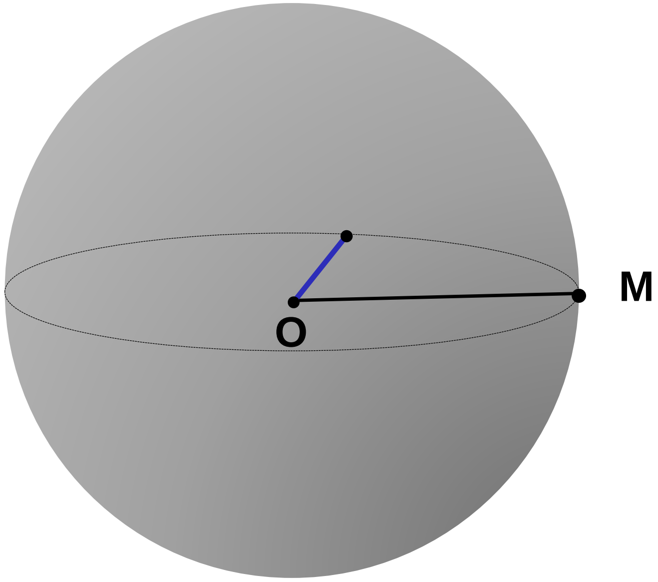

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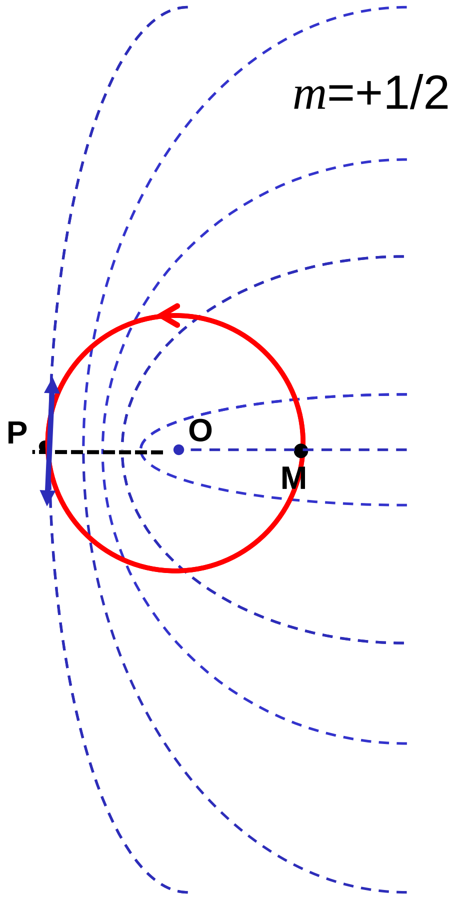

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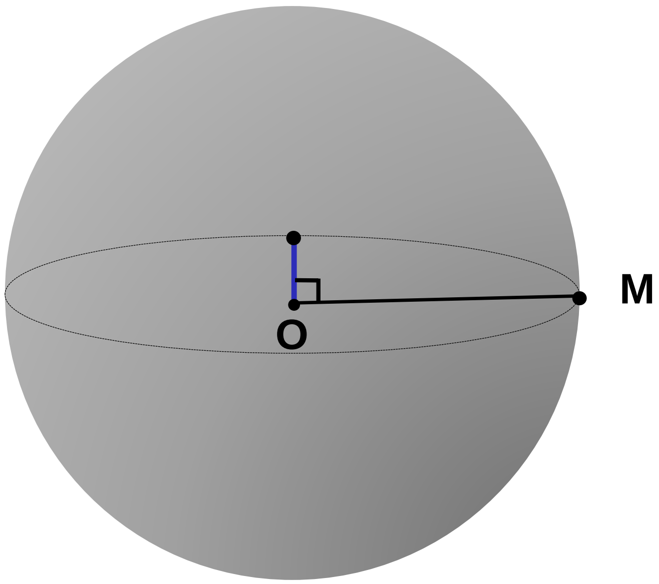

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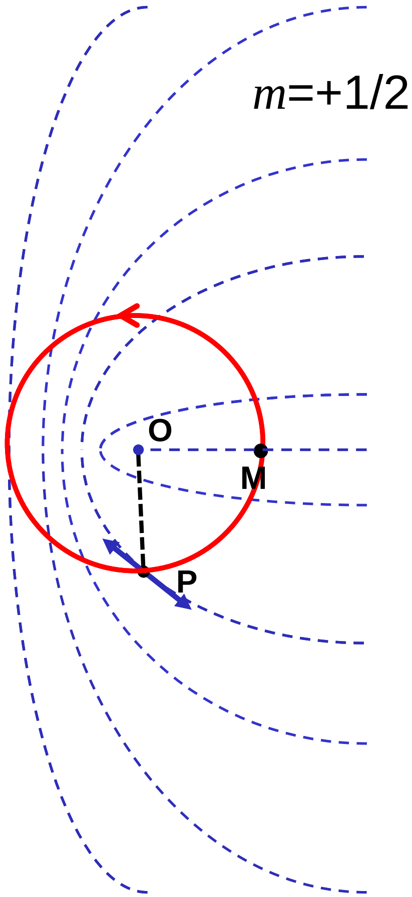

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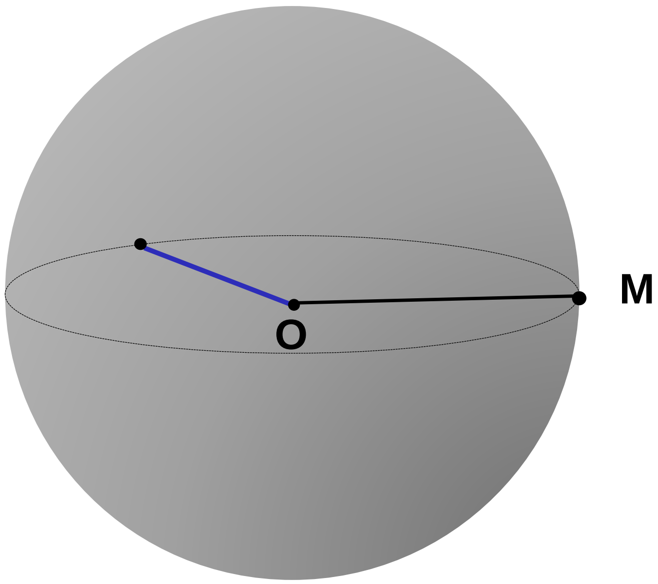

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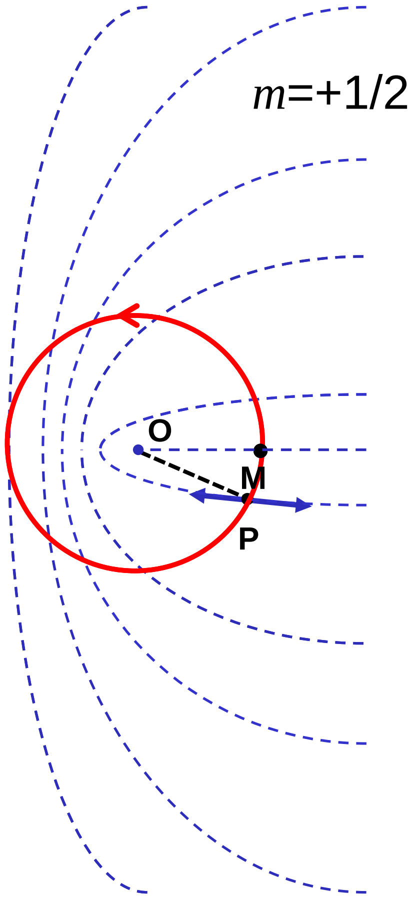

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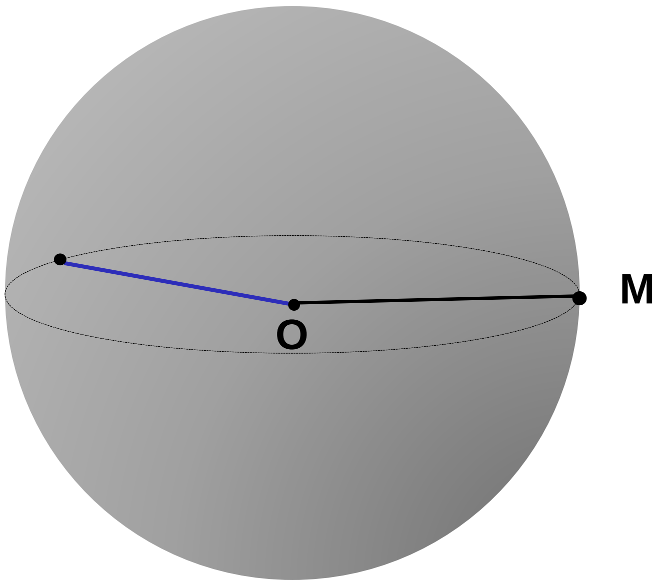

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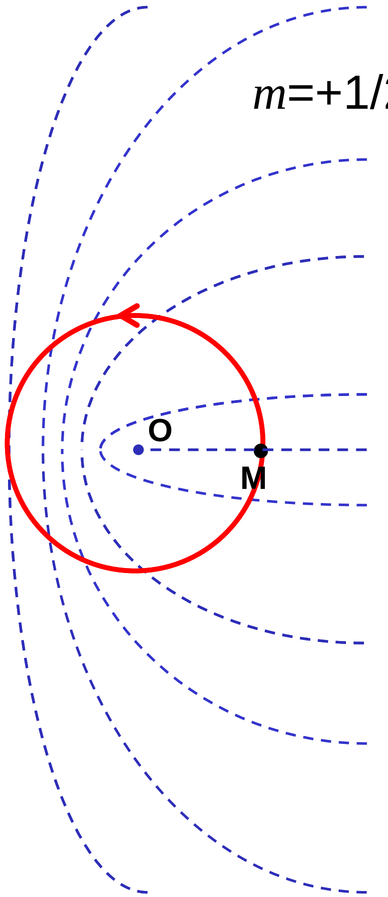

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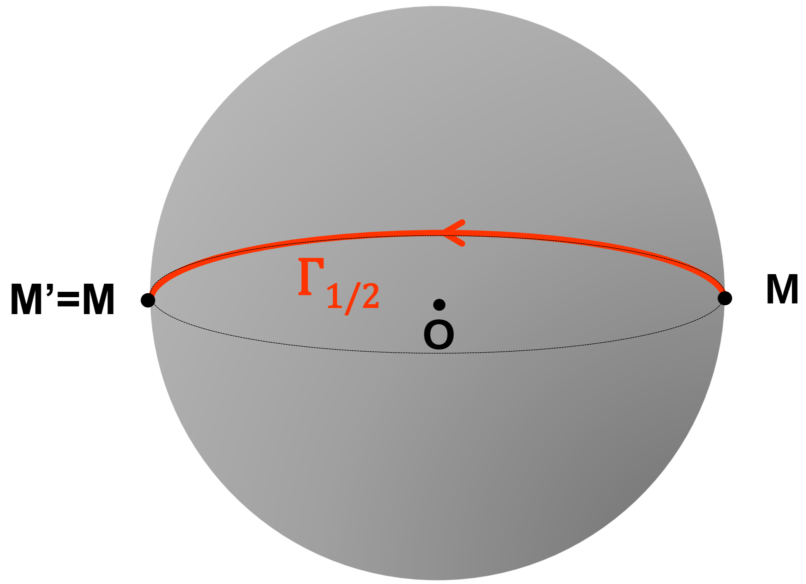
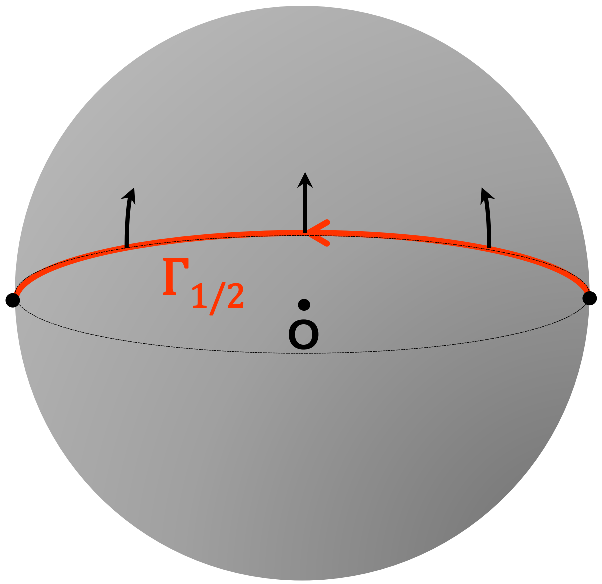


The homotopy loop cannot be shrunk to a point
= no escape in the 3rd dimension = **topologically non-removable defect**.

Moebius disclinations

$$\pi_1(\mathbb{R}P^2) = Z_2 = \{0, 1\}$$



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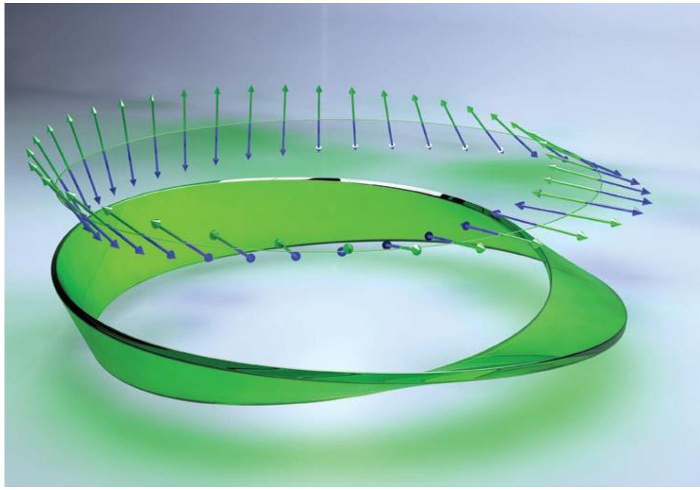
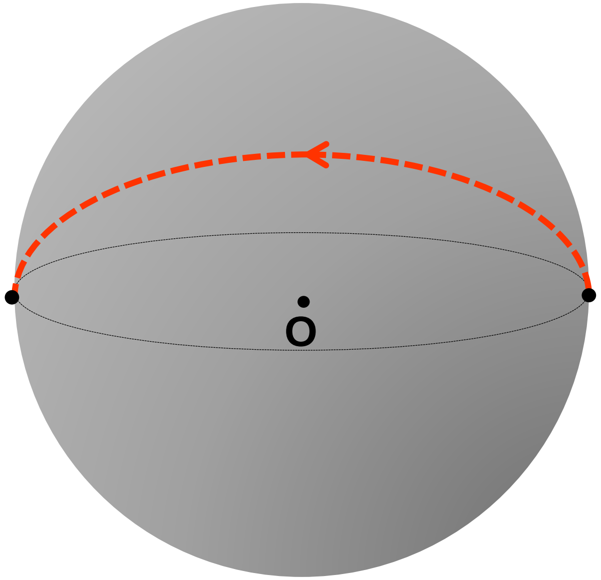


Moebius ribbon



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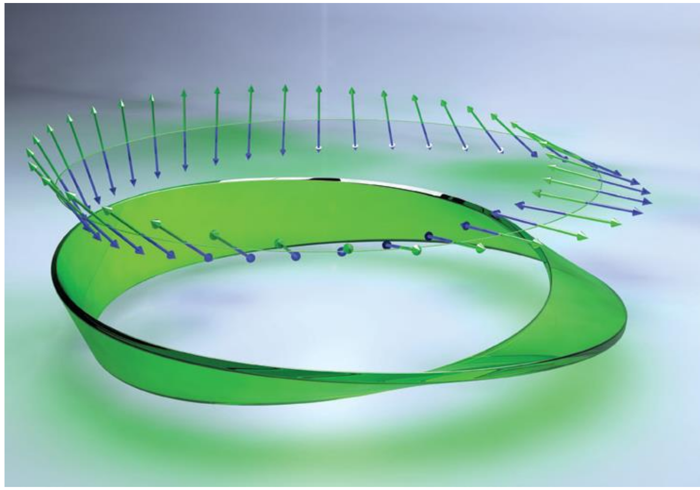

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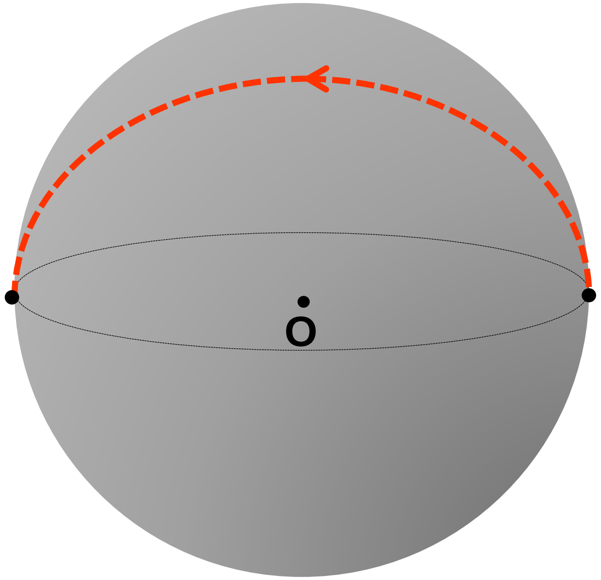

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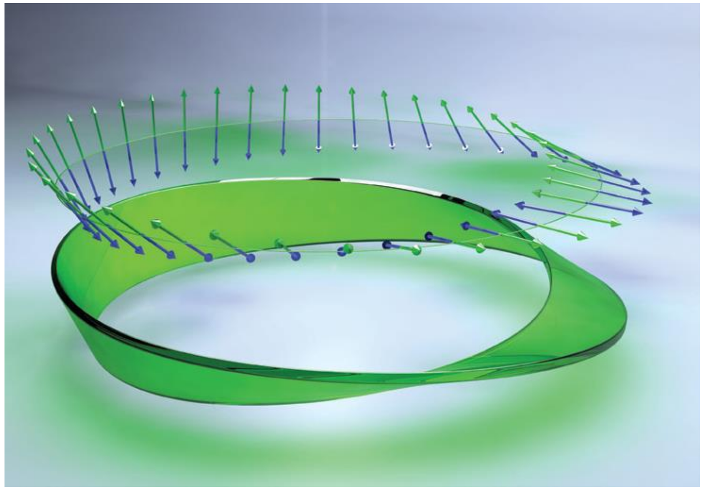

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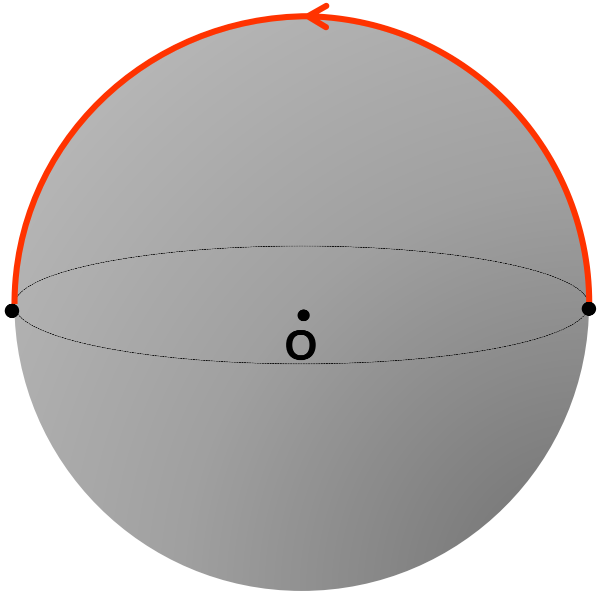

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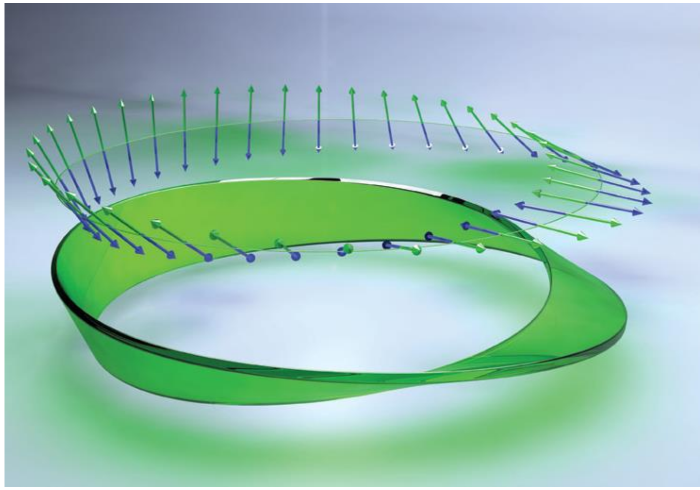

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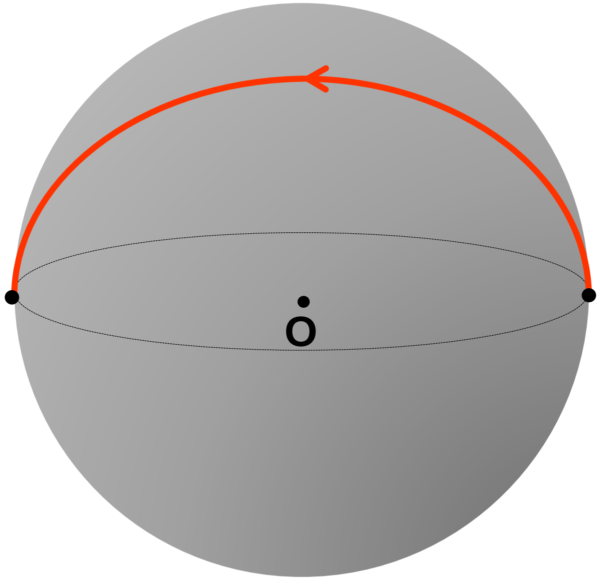

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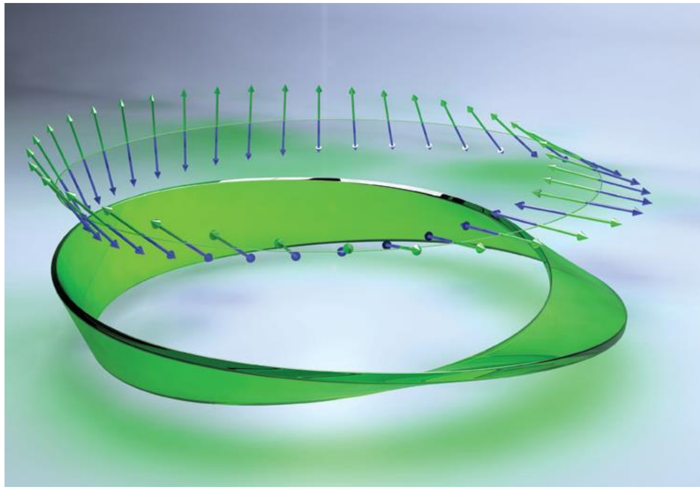

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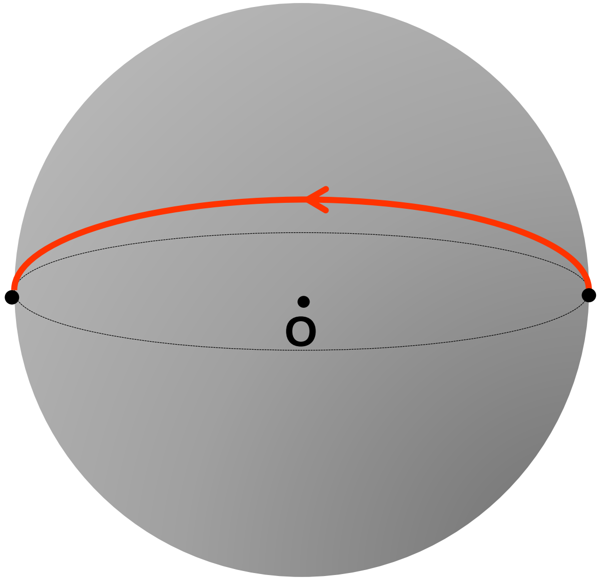

Moebius disclinations

AA Balinskii, GE Volovik, EI Kats. Sov. Phys. JETP 60 (1984)

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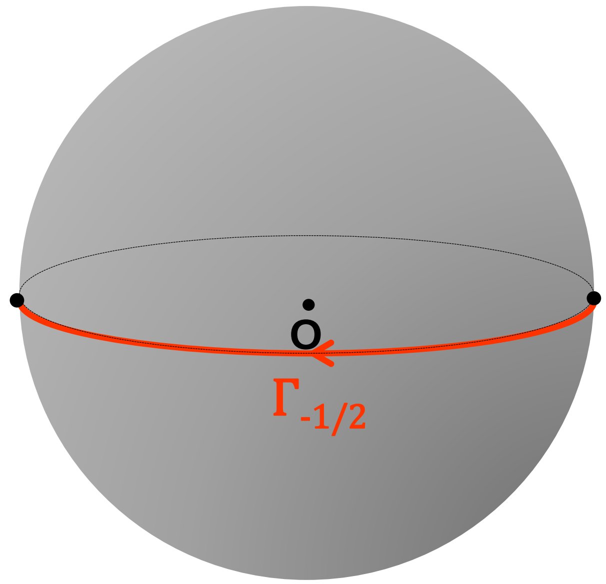

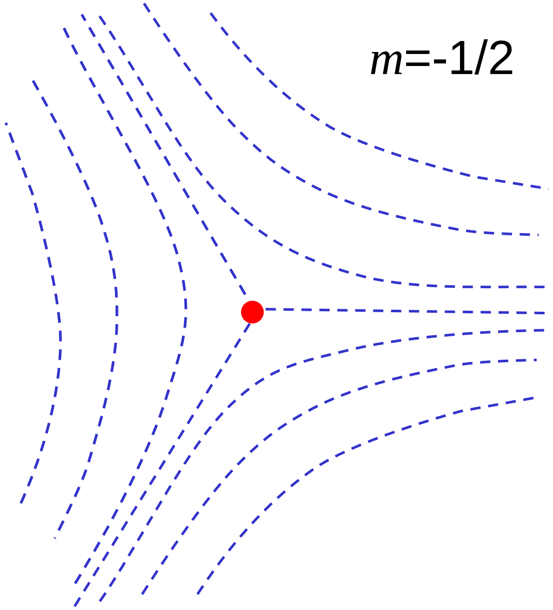

$\Gamma_{1/2}$ and $\Gamma_{-1/2}$ are topologically equivalent.

⇒ Disclinations of half-integer strengths belong to the same homotopy class N= 1

Solution to the riddle

⇒ Although *m* is called the « topological charge » of the defect, it is the absolute value of *m* that matters for topology.

$$N = 1 - E\left(|m|\right)$$

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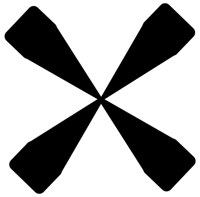
- Polarising microscopy reveals « Schlieren patterns », which depend on this topological invariant, as the number of dark brushes = $4 \times |m|$.

$m = \pm 1/2$

$m = \pm 1$



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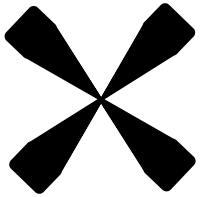
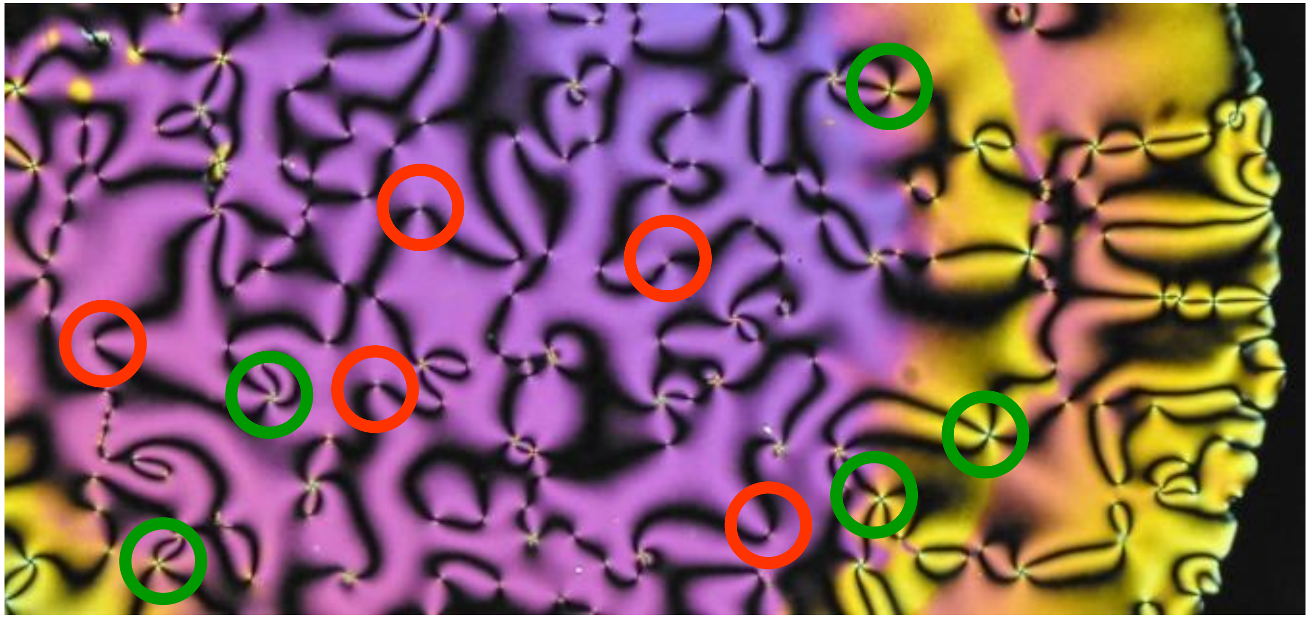


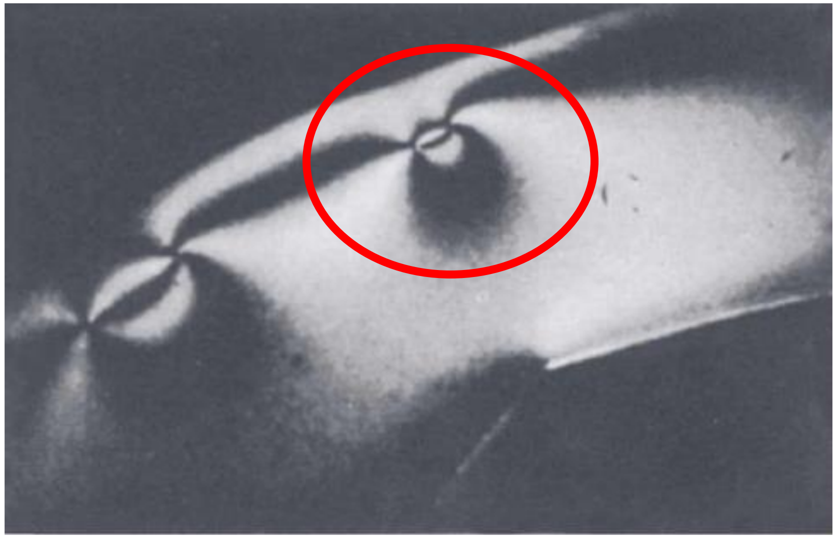


Algebra of linear defects

S Chandrasekhar. Liquid crystals (1984)

- The set is the quotient group $\mathbb{Z}/2\mathbb{Z}$={0,1} with a law of addition +.

0+0=0 0+1=1 1+1=0



Analogously to electrostatics, defects of opposite charges attract each other (repell otherwise):

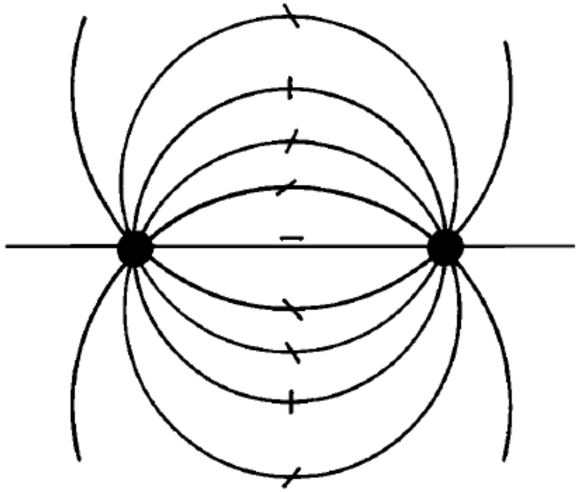


Fig. 3.5.3. Curves of equal alignment around a pair of singularities of equal and opposite strengths. The orientations marked on the circles refer to the case $s = 1$, $c = 0$.

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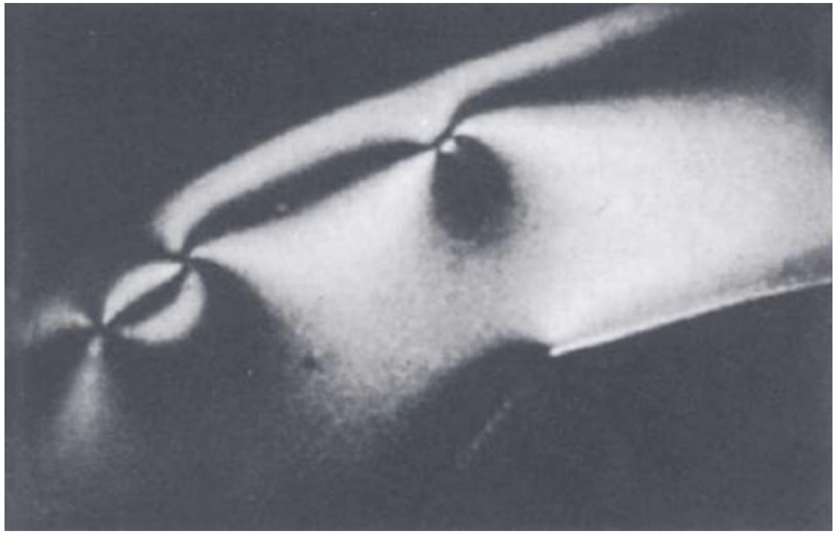

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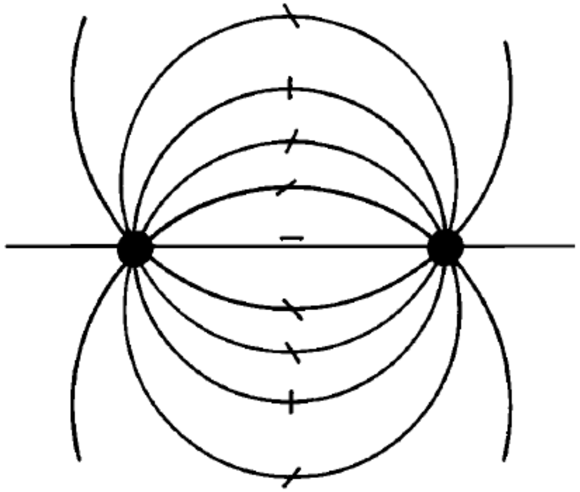

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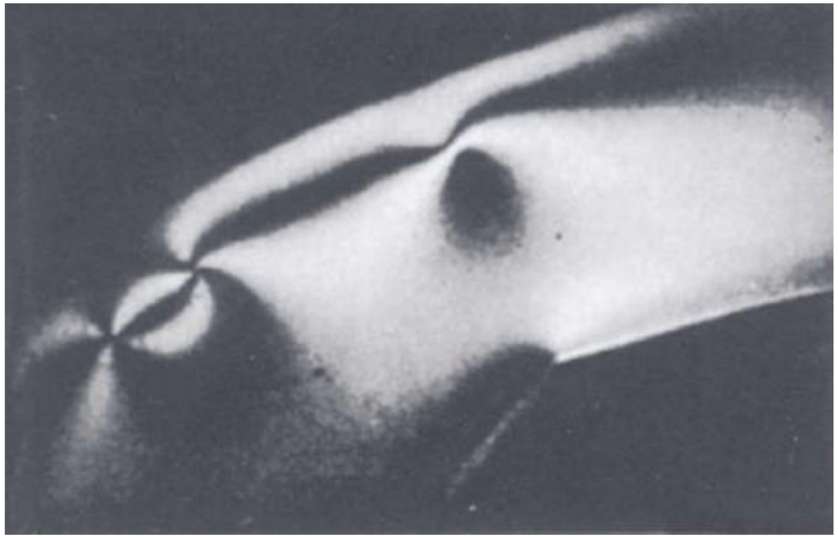

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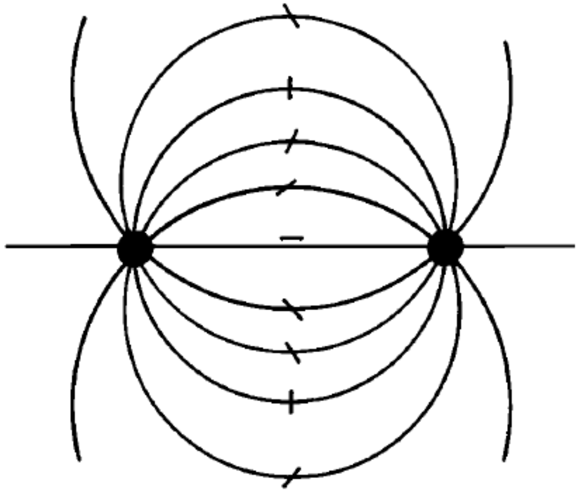

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C Zhang. PhD thesis. Carnegie Mellon university (2017)

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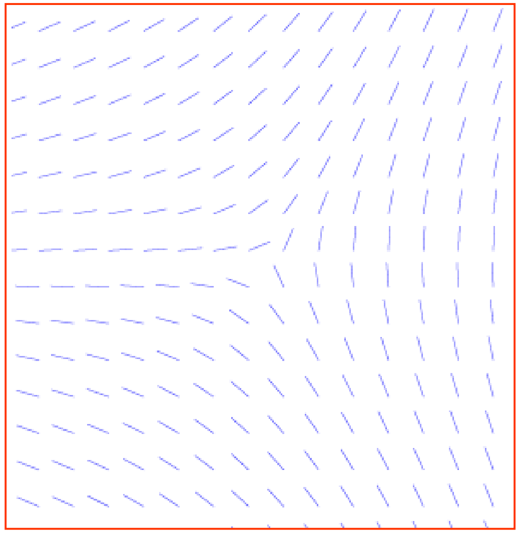
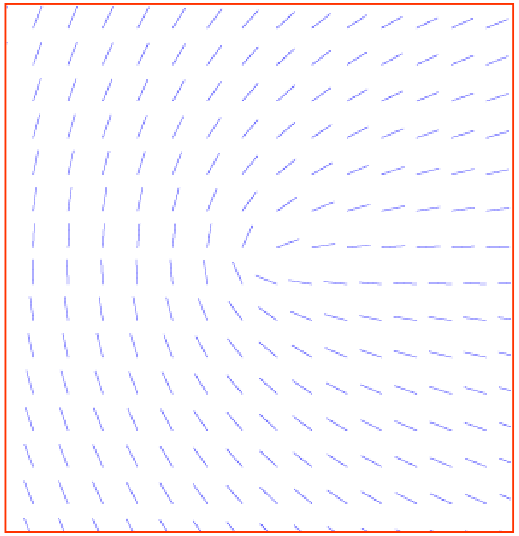


m=-1/2



m=+1/2

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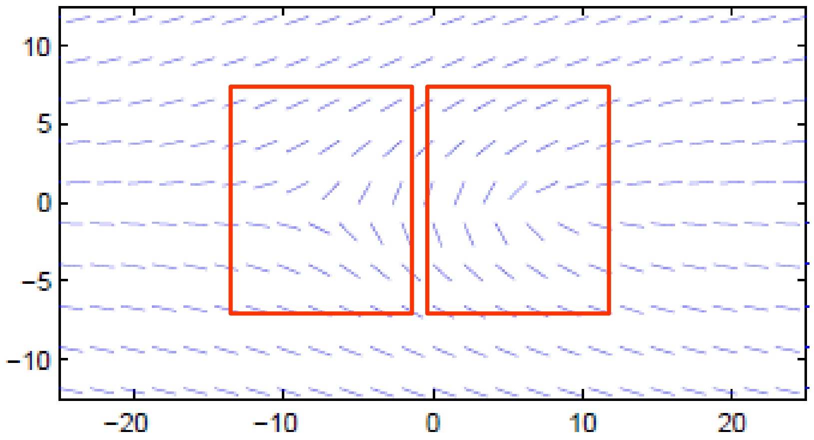

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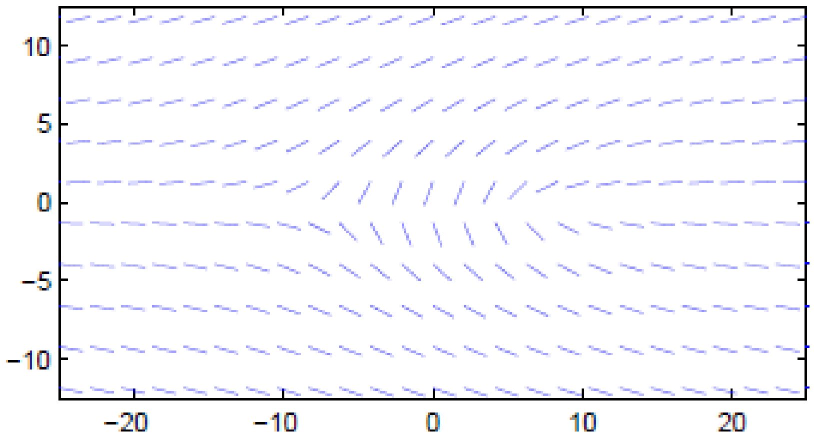
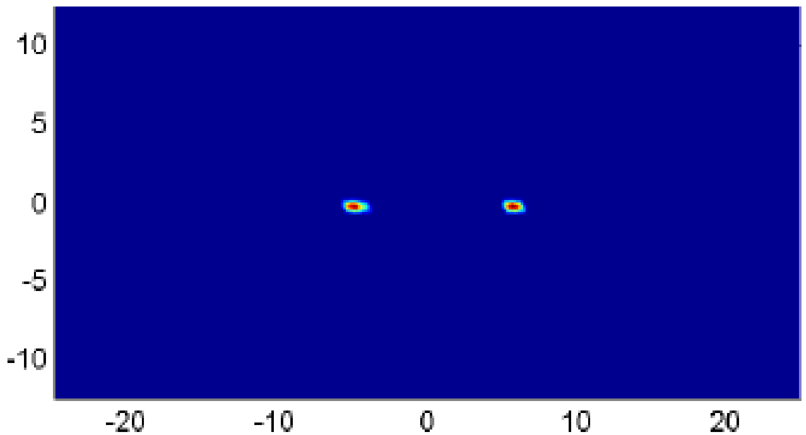


(a) Director snapshot at t=0.



(b) Energy density plot at t=0.

Algebra of linear defects

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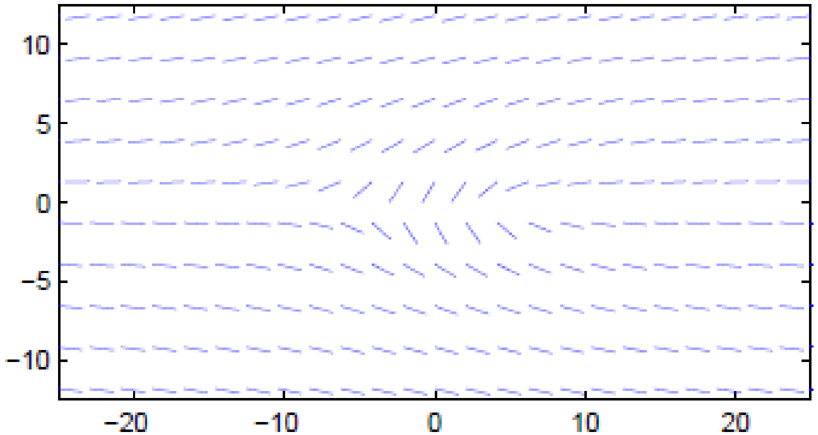
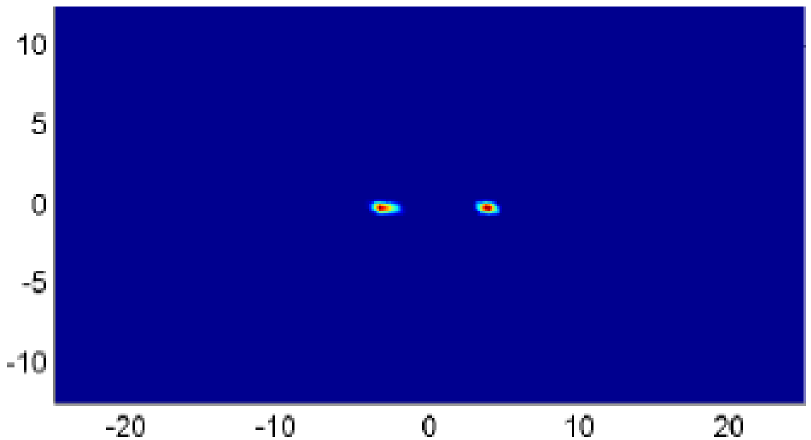


(c) Director snapshot at t=1.



(d) Energy density plot at t=1.

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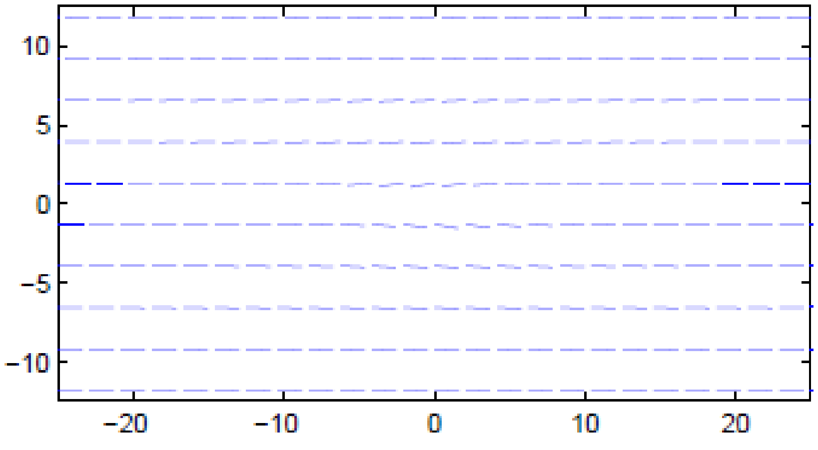
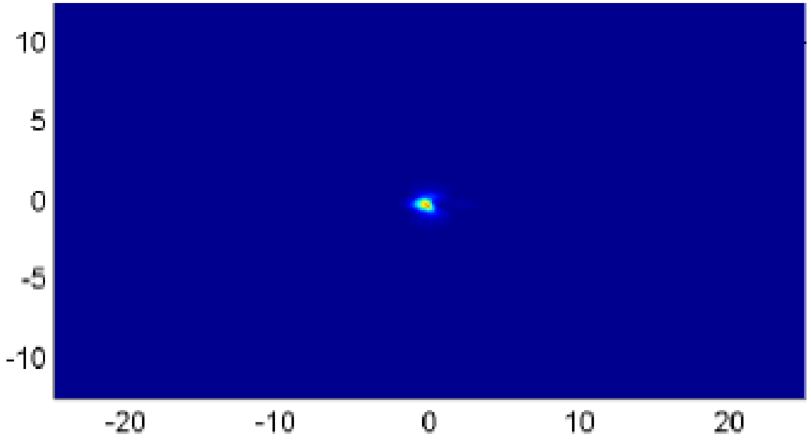


(e) Director snapshot at t=1.2.



(f) Energy density plot at t=1.2.

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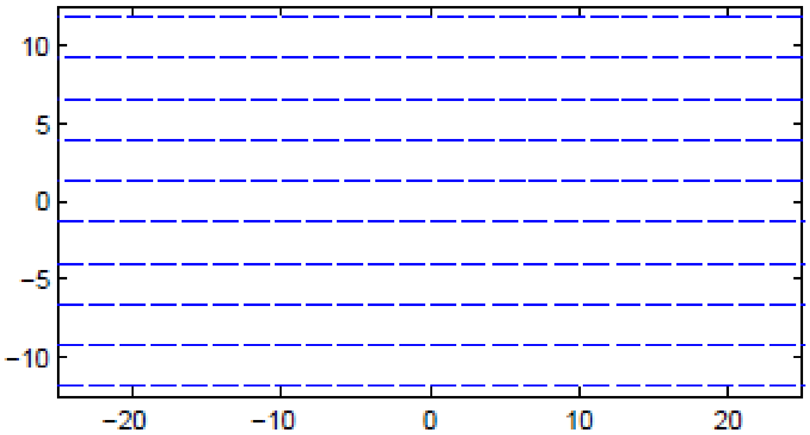
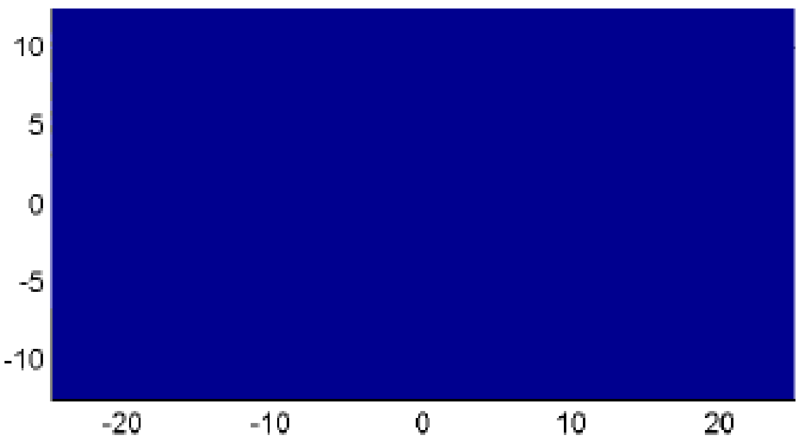


(g) Director snapshot at t=1.25.

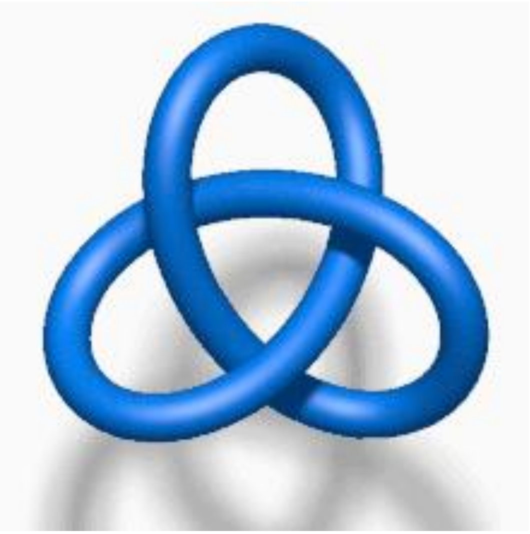


(h) Energy density plot at t=1.25.

Other topological numbers

- **Is $|m|$ enough to characterize the topology of a line defect ?**

Locally yes, but globally no, as a line defect can self-connect, entangle with itself (« nematic braids ») or more…

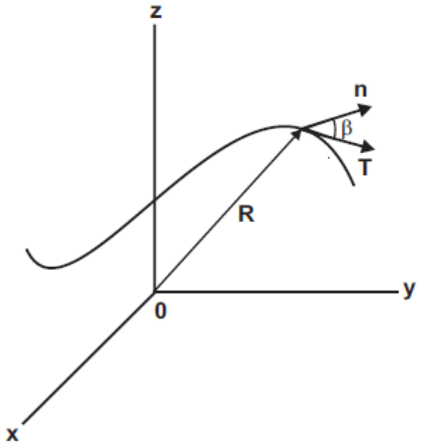


To go further: self-linking number, Jänich's index, Poincaré-Hopf's index…

Fermat-Grandjean principle (1919)

- Extraordinary light paths obey a least action principle

$$\delta\left[\int_A^B N_e(\boldsymbol{r})\,dl\right]=0=\delta\left[\int_A^B \sqrt{\varepsilon_\perp \cos^2\beta(\boldsymbol{r})+\varepsilon_{||}\sin^2\beta(\boldsymbol{r})}\;dl\right]$$



Fermat-Grandjean principle (1919)

C Satiro, F Moraes. EPJ E 20 (2006)

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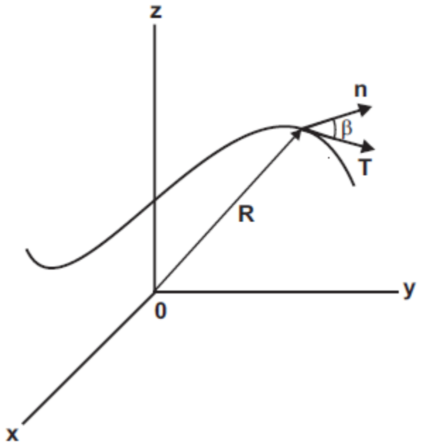

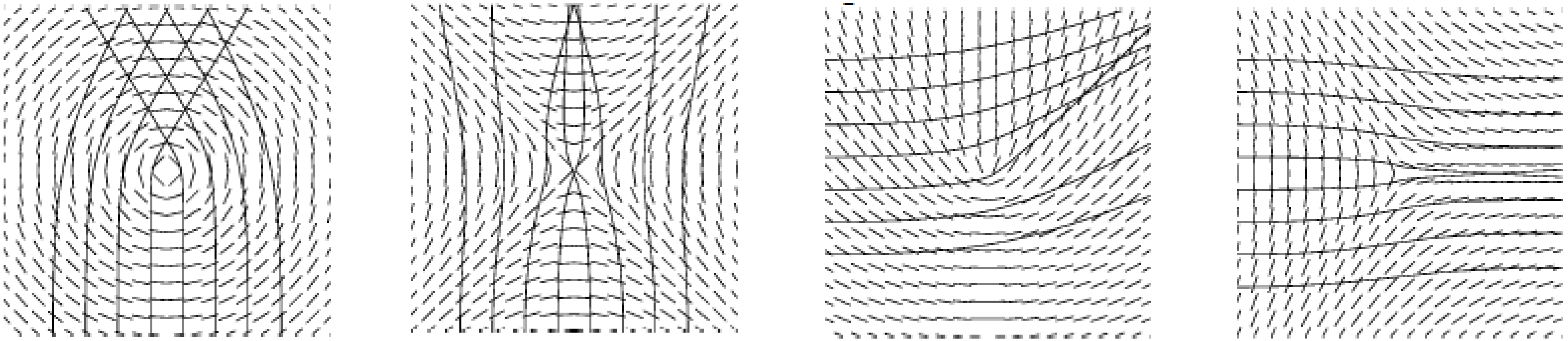

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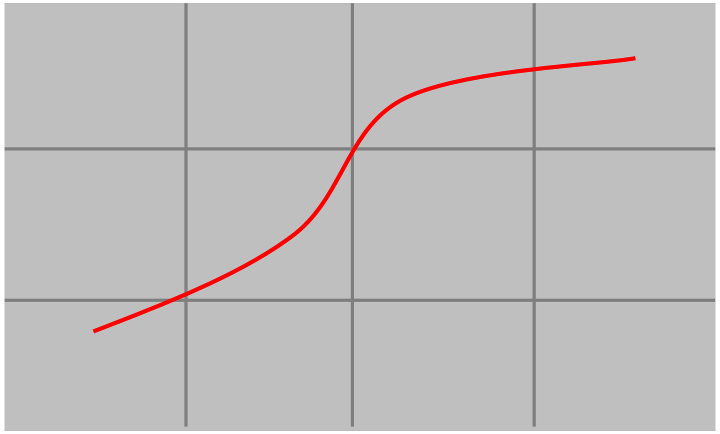
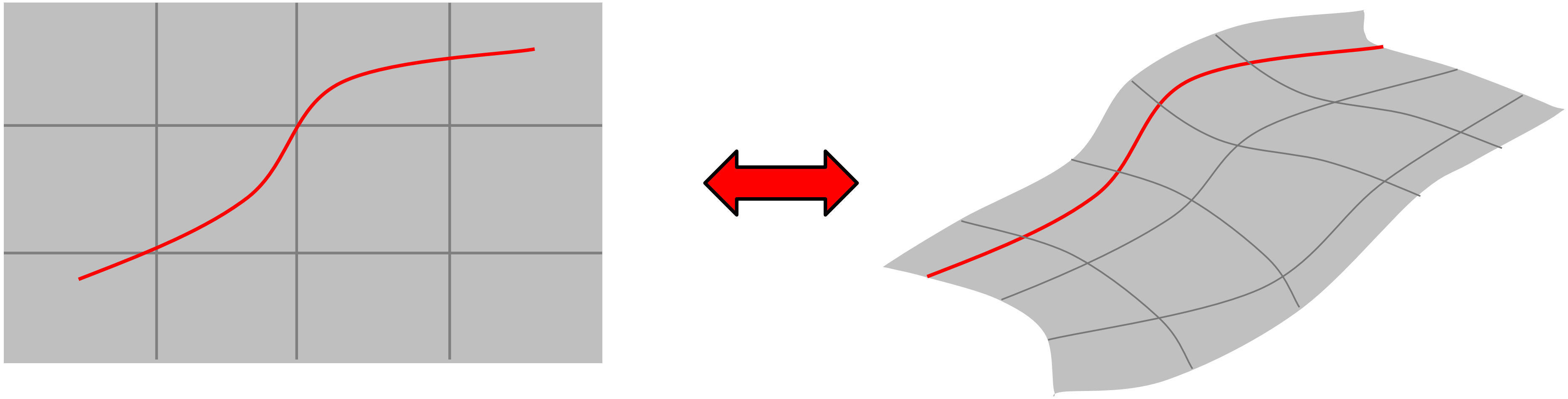


Curved path in a Euclidean space

Fermat-Grandjean principle (1919)

- Extraordinary light paths obey a least action principle

$$\delta\left[\int_A^B N_e(\boldsymbol{r})\,dl\right] = 0 = \delta\left[\int_A^B \sqrt{\varepsilon_\perp \cos^2\beta(\boldsymbol{r}) + \varepsilon_\parallel \sin^2\beta(\boldsymbol{r})}\;dl\right]$$



Curved path in a Euclidean space *« Straight path » in a curved space*

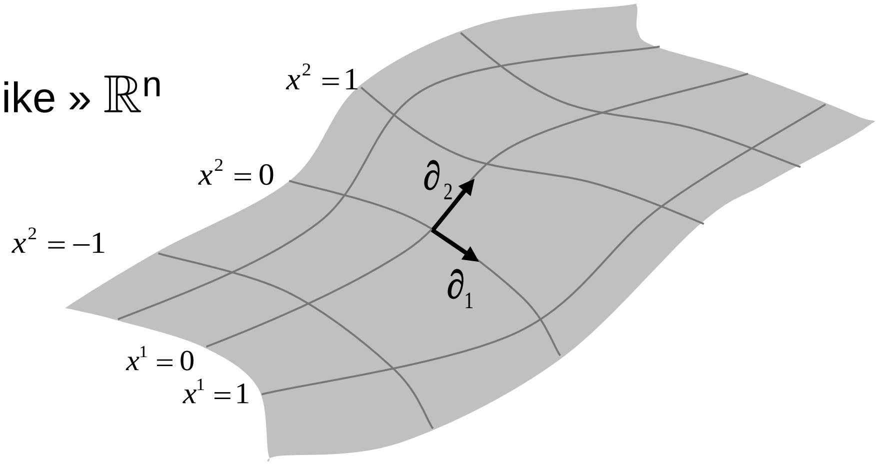
Reminder of Riemann geometry

▶ How to describe a curved geometry ?

- A **n-manifold** = smooth hypersurface that locally « looks like » $\mathbb{R}^n$

  - *coordinate system* x^μ

  - *coordinate basis* ∂_μ



- A **metric structure** to measure lengths

$$g_{\mu\nu} = \partial_\mu . \partial_\nu$$

$$ds^2 = d\mathbf{x}.d\mathbf{x} = \left(dx^\mu \partial_\mu\right).\left(dx^\nu \partial_\nu\right) = dx^\mu \, dx^\nu g_{\mu\nu}$$

Generalized Pythagoras' theorem

- A **connection** to perform covariant differentiation… $\nabla_\alpha V^\mu = \partial_\alpha V^\mu + \Gamma^\mu_{\alpha\nu} V^\nu$

Levi-Civita connection

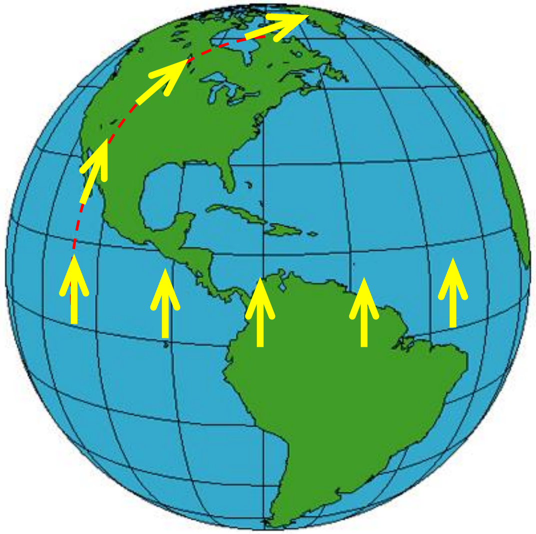
Reminder of Riemann geometry

… and to *parallel-transport* vectors, tensors, spinors …

$$\frac{DT^{\mu..}{}_{\nu..}}{d\lambda} = \frac{dx^{\alpha}}{d\lambda}\nabla_{\alpha}T^{\mu..}{}_{\nu..} = 0$$

$$= \frac{dT^{\mu..}{}_{\nu..}}{d\lambda} + \Gamma^{\mu}_{\alpha\beta}T^{\beta..}{}_{\nu..}\frac{dx^{\alpha}}{d\lambda} + ... - \Gamma^{\beta}_{\alpha\nu}T^{\mu..}{}_{\beta..}\frac{dx^{\alpha}}{d\lambda} - ...$$

Searching for the North pole with a compass



Reminder of Riemann geometry

… and to *parallel-transport* vectors, tensors, spinors …

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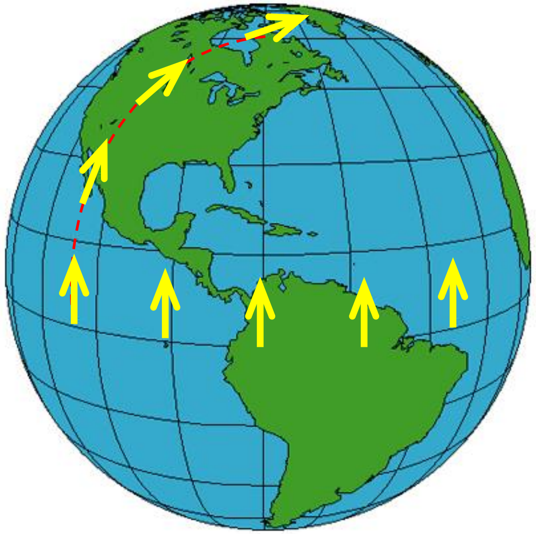


$$\frac{DT^{\mu..}{}_{\nu..}}{d\lambda} = \frac{dx^\alpha}{d\lambda}\nabla_\alpha T^{\mu..}{}_{\nu..} = 0$$

$$= \frac{dT^{\mu..}{}_{\nu..}}{d\lambda} + \Gamma^\mu_{\alpha\beta} T^{\beta..}{}_{\nu..}\frac{dx^\alpha}{d\lambda} + \ldots - \Gamma^\beta_{\alpha\nu} T^{\mu..}{}_{\beta..}\frac{dx^\alpha}{d\lambda} - \ldots$$

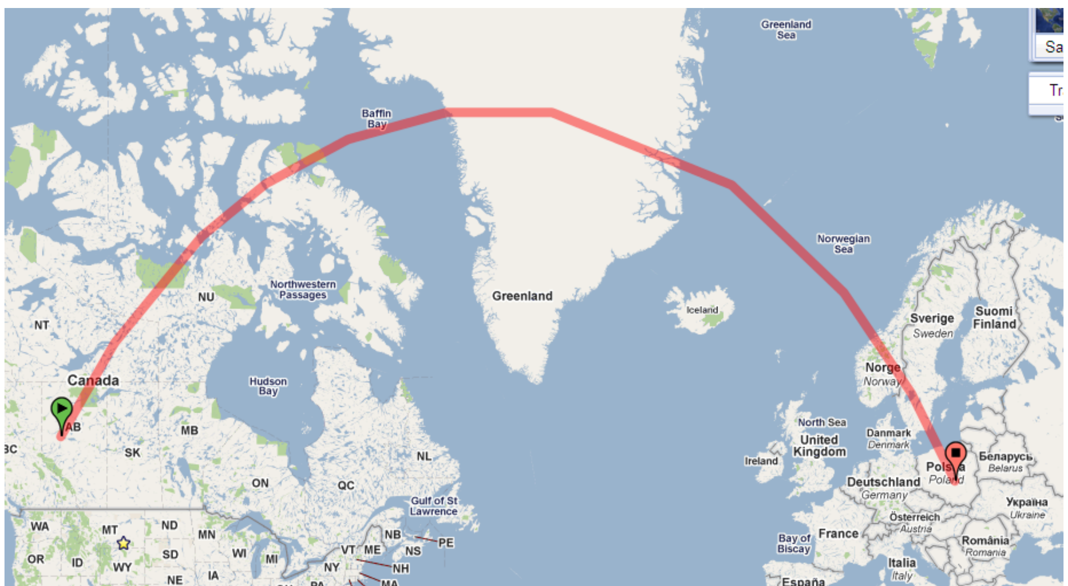
- Parallel transport can be used to define a special class of curves, the **geodesics**, which are the curved-geometry generalizations of the Euclidean notion of straight lines. A geodesic curve is one that parallel-transports its own tangent vector (= autoparallel curve):

$$\frac{D}{d\lambda}\left(\frac{dx^\mu}{d\lambda}\right) = 0 = \frac{dx^\alpha}{d\lambda}\left(\frac{\partial}{\partial x^\alpha}\left[\frac{dx^\mu}{d\lambda}\right] + \Gamma^\mu_{\alpha\nu}\frac{dx^\nu}{d\lambda}\right) \quad \boxed{\Rightarrow 0 = \frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\mu_{\alpha\nu}\frac{dx^\nu}{d\lambda}\frac{dx^\alpha}{d\lambda}}$$

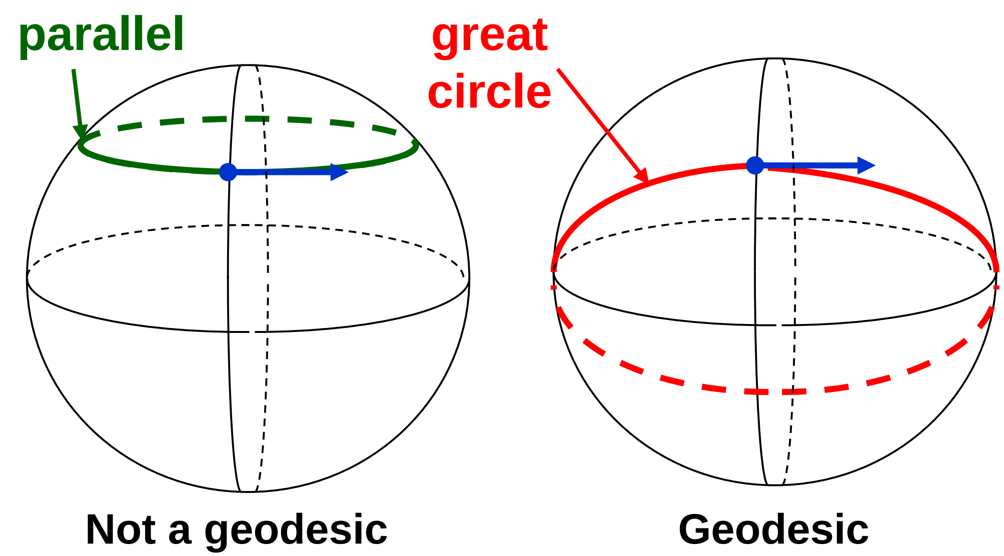
↔ *Newton's 2nd law*

Reminder of Riemann geometry

- But most importantly, geodesics are the curves of extremal lengths (cf. Fermat-Grandjean principe). A warning: in the presence of curvature, actual geodesics may be very counter-intuitive:

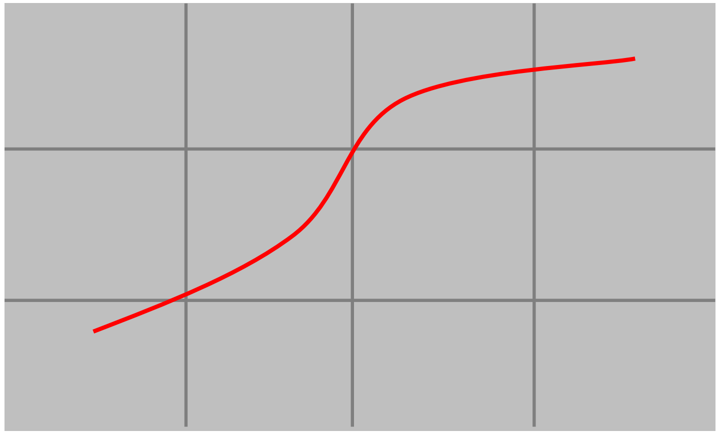


Shortest path between Calgary and Warsaw (flight plan)

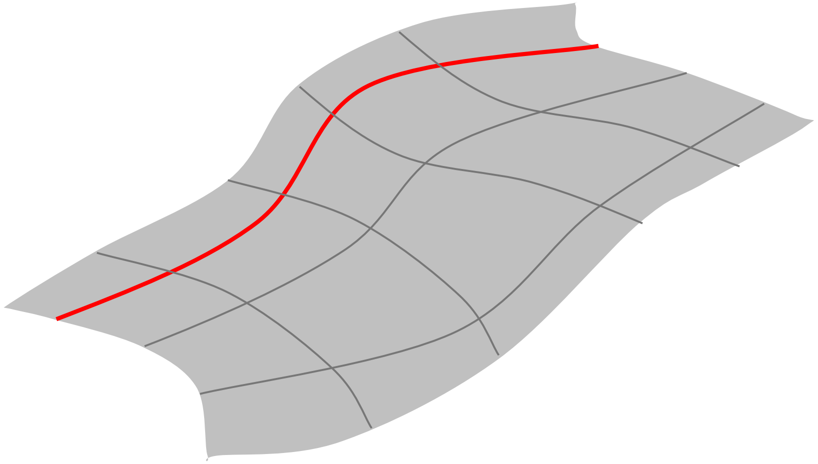


Transformation optics

U Leonhardt,TG. Philbin. Progress in Optics 53 (2009)
H Chen et al. Nature materials, 9 (2010)



Light path inside matter in Euclidean space

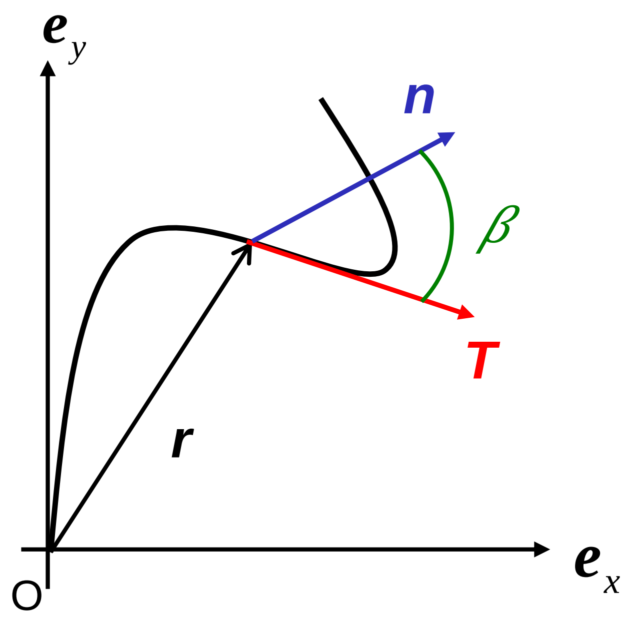


Geodesic in vacuum in curved space

$$ds^2_{2D} = N_e^2(\boldsymbol{r})\,dl^2 \quad \Leftrightarrow \quad ds^2_{2D} = g_{ij}dx^i dx^j$$

⇨ How to find the metric tensor representing a defective liquid crystal ?

Recipe for the line element



1. *Express the tangent vector T (planar path)*

$$\boldsymbol{r} = r\cos\theta\boldsymbol{e}_x + r\sin\theta\boldsymbol{e}_y$$

$$\boldsymbol{T} = \frac{d\boldsymbol{r}}{dl} = \left(\dot{r}\cos\theta - r\dot{\theta}\sin\theta\right)\boldsymbol{e}_x + \left(\dot{r}\sin\theta + r\dot{\theta}\cos\theta\right)\boldsymbol{e}_y$$

2. *Write the components of the director field* $\boldsymbol{n} = \cos\psi\boldsymbol{e}_x + \sin\psi\boldsymbol{e}_x$

$$\Rightarrow \cos\beta = \boldsymbol{n}.\boldsymbol{T} = \dot{r}\cos(\psi-\theta) + r\dot{\theta}\sin(\psi-\theta)$$

$$\sin\beta = \left\|\boldsymbol{n}\times\boldsymbol{T}\right\| = -\dot{r}\sin(\psi-\theta) + r\dot{\theta}\cos(\psi-\theta)$$

3. *Replace in the Fermat-Grandjean line element and see the magic*

$$N_e^2(\boldsymbol{r})dl^2 = \left(\varepsilon_\perp\cos^2\beta(\boldsymbol{r}) + \varepsilon_{||}\sin^2\beta(\boldsymbol{r})\right)dl^2$$

Recipe for the line element

- To spare repelling calculations, one sticks to $m=1,\ \psi_0=0$, but the proof is in the same fashion for the general case.

$$\Rightarrow \cos\beta = \dot{r}\cos(\psi-\theta) + r\dot{\theta}\sin(\psi-\theta) = \dot{r}$$

$$\sin\beta = -\dot{r}\sin(\psi-\theta) + r\dot{\theta}\cos(\psi-\theta) = r\dot{\theta}$$

$$\Rightarrow ds_{2D}^2 = N_e^2(\boldsymbol{r})\,dl^2 = \left(\varepsilon_\perp \dot{r}^2 + \varepsilon_\parallel r^2\dot{\theta}^2\right)dl^2 = \left(\varepsilon_\perp\left[\frac{dr}{dl}\right]^2 + \varepsilon_\parallel r^2\left[\frac{d\theta}{dl}\right]^2\right)dl^2 = \varepsilon_\perp dr^2 + \varepsilon_\parallel r^2 d\theta^2$$

A simple rescaling on the radial coordinate finally leads to $\boxed{ds_{2D}^2 = d\rho^2 + \alpha^2\rho^2 d\theta^2}$

Recipe for the line element

- To spare repelling calculations, one sticks to $m = 1,\ \psi_0 = 0$, but the proof is in the same fashion for the general case.

$$\Rightarrow \cos\beta = \dot{r}\cos(\psi - \theta) + r\dot{\theta}\sin(\psi - \theta) = \dot{r}$$
$$\sin\beta = -\dot{r}\sin(\psi - \theta) + r\dot{\theta}\cos(\psi - \theta) = r\dot{\theta}$$

$$\Rightarrow ds_{2D}^2 = N_e^2(\boldsymbol{r})\,dl^2 = \left(\varepsilon_\perp \dot{r}^2 + \varepsilon_\parallel r^2\dot{\theta}^2\right)dl^2 = \left(\varepsilon_\perp\left[\frac{dr}{dl}\right]^2 + \varepsilon_\parallel r^2\left[\frac{d\theta}{dl}\right]^2\right)dl^2 = \varepsilon_\perp dr^2 + \varepsilon_\parallel r^2 d\theta^2$$

A simple rescaling on the radial coordinate finally leads to $\boxed{ds_{3D}^2 = d\rho^2 + \alpha^2\rho^2 d\theta^2 + dz^2}$

⇨ *What kind of geometry does this represent?*

What a wedge cut does to space

- Ricci curvature scalar: $R(\rho)=\frac{(1-\alpha)}{\alpha\rho}\delta(\rho)$ = flat everywhere but on the z-axis.

  For a circle of unit radius about the z-axis, the perimeter is given by $p=\oint_{\rho=1} ds_{3D}=\alpha\oint d\theta=2\pi\alpha$

⇒ « conical » geometry

What a wedge cut does to space

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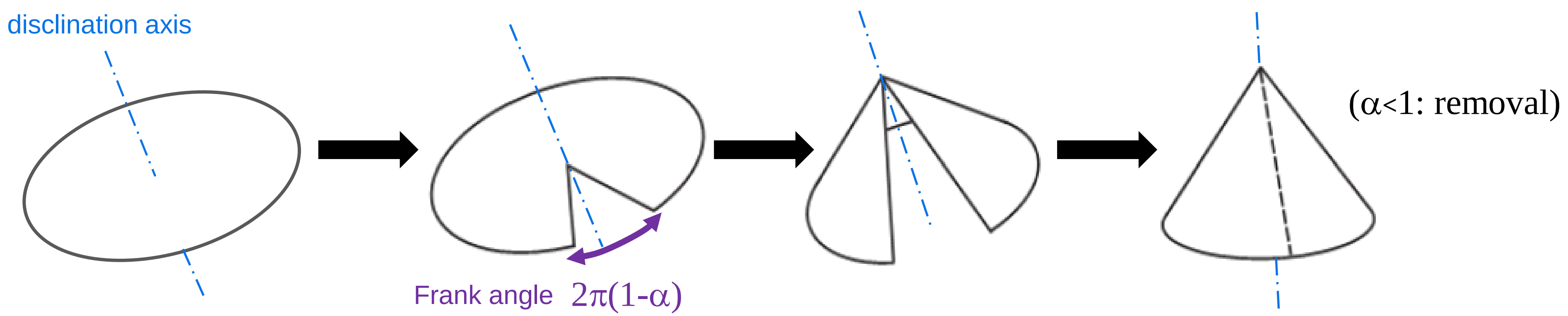
⇒ « conical » geometry

- Volterra process for a deficit-angle or wedge disclination:

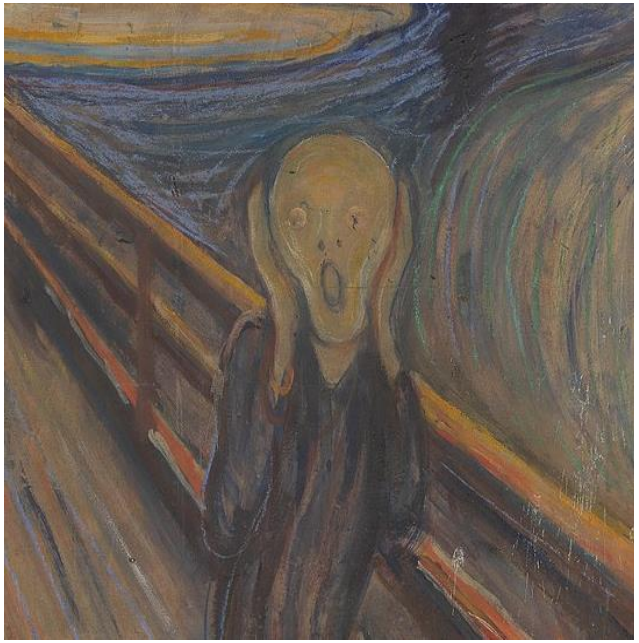


Just for fun…

- The general line element for a straight disclination of any topological charge is

$$ds_{3D}^2 = \left(\varepsilon_\perp \cos^2\left[(m-1)\theta+\psi_0\right]+\varepsilon_\parallel \sin^2\left[(m-1)\theta+\psi_0\right]\right)dr^2$$
$$+\left(\varepsilon_\perp \sin^2\left[(m-1)\theta+\psi_0\right]+\varepsilon_\parallel \cos^2\left[(m-1)\theta+\psi_0\right]\right)r^2 d\theta^2$$
$$-\left(\varepsilon_\parallel-\varepsilon_\perp\right)\sin\left[2(m-1)\theta+2\psi_0\right]rdrd\theta+dz^2$$

What should be done next involves computing the connection coefficients, the Riemann curvature tensor…



Distribution of defects

S Fumeron et al. Eur. Phys. J. B 90 (2017)

- In real-life, defects are not isolated. Sometimes, it is possible to find analytical expressions for the line element:

Discrete distribution of disclinations $$ds^2 = -c^2dt^2 + e^{-4V(x,y)}\left(dx^2 + dy^2\right) + dz^2$$

$$V(x,y) = \frac{|F|}{4\pi}\ln\left[\left(\frac{\cosh^2\left(\frac{\pi}{2a}(y-b)\right) - \cos^2\left(\frac{\pi x}{2a}\right)}{\cosh^2\left(\frac{\pi}{2a}(y-b)\right) - \sin^2\left(\frac{\pi x}{2a}\right)}\right)\left(\frac{\cosh^2\left(\frac{\pi}{2a}(y+b)\right) - \sin^2\left(\frac{\pi x}{2a}\right)}{\cosh^2\left(\frac{\pi}{2a}(y+b)\right) - \cos^2\left(\frac{\pi x}{2a}\right)}\right)\right]$$

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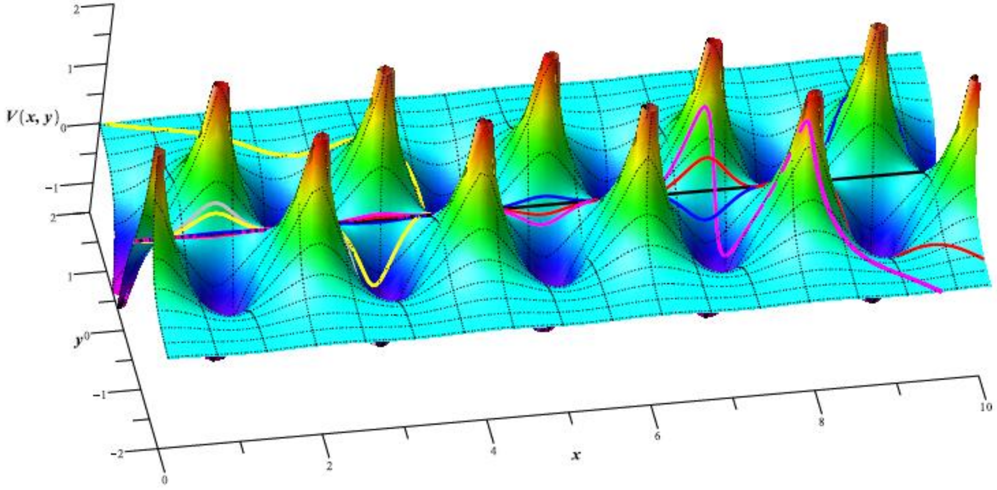
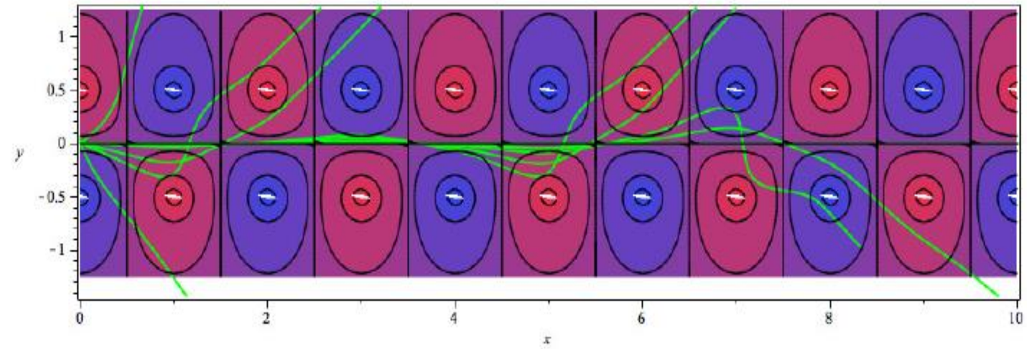
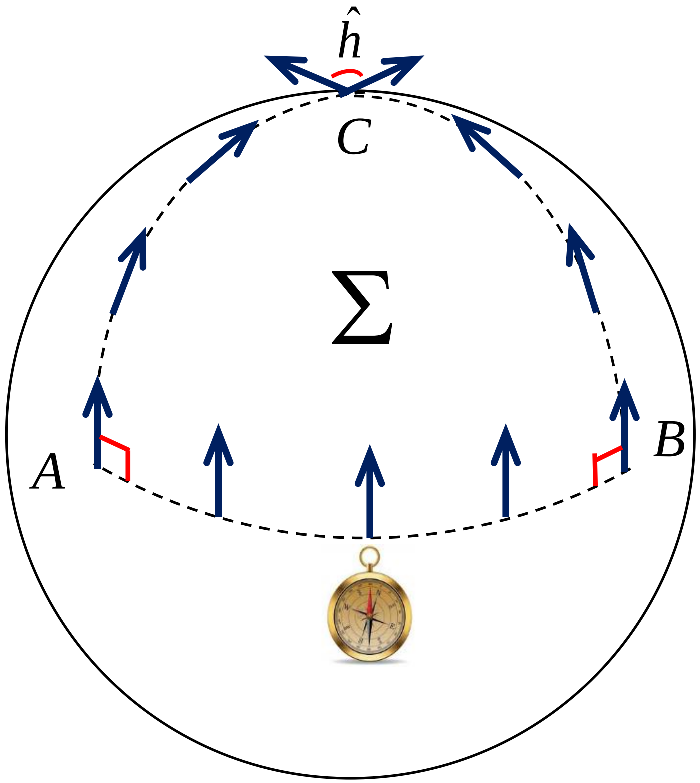




Holonomy: intuitive approach

- Let us go back to « lost traveller problem »: after a closed loop, a parallel-transported vector fails to recover its initial direction: this is called (an)**holonomy**. How to understand that ?



Searching for the North pole

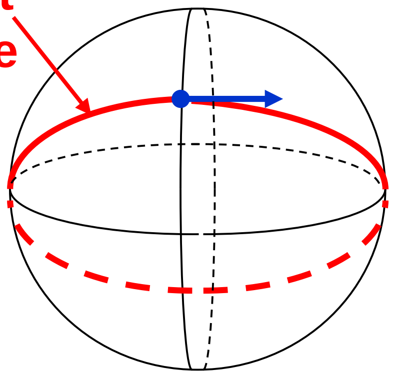
⇨ **Girard's formula** $\Sigma = R^2\Omega = R^2\left(\hat{A}+\hat{B}+\hat{C}-\pi\right)$

Here, this simplifies into $F\Sigma = \hat{C}$, which also turns out to be $\hat{h}$, the mismatch angle.

Hence, the mismatch angle is a measure of the Gaussian curvature *F* of the surface Σ bounded by the closed circuit.

This result is an basic outcome of the **Ambrose-Singer theorem**: for a given connection on a vector bundle, the curvature corresponds to the surface density of holonomy.

Holonomy: intuitive approach



$$F \neq 0, \kappa_g = 0$$

$$F = 0, \kappa_g \neq 0$$

- But the Gaussian curvature *F* is also related to the topology of the surface:

Gauss-Bonnet theorem

$$\iint_{\Sigma} F dS + \oint_{\partial\Sigma} \kappa_g ds = 2\pi\chi$$

Gauss curvature (local) ↔ *GEOMETRY*

Geodesic curvature (local) ↔ *GEOMETRY*

Euler characteristic (global) ↔ *TOPOLOGY*

$$\Rightarrow \hat{h} = 2\pi\chi - \oint_{\partial\Sigma} \kappa_g ds$$

Therefore, holonomy is also connected to topology.

Holonomy: formal approach

S Carroll. Spacetime and geometry (2003)

H Hertz

- More generally, one defines **holonomy** as the failure to transport any information (such as the orientation of a vector) on a closed circuit on a curved surface Σ.
- Let there be a path parametrized by λ along which a vector V is parallel-transported. The *parallel propagator* Π is defined as

E Cartan

$$V^{\mu}(\lambda)=\Pi^{\mu}{}_{\rho}(\lambda)V^{\rho}(0)$$

- But the transport parallel condition also writes as: $\dfrac{DV^{\nu}}{d\lambda}=0\Rightarrow\dfrac{dV^{\nu}}{d\lambda}=-\Gamma^{\nu}_{\sigma\mu}\dfrac{dx^{\sigma}}{d\lambda}V^{\mu}=A^{\nu}{}_{\mu}V^{\mu}$

Therefore the parallel propagator obeys

$$\frac{d}{d\lambda}\Pi^{\nu}{}_{\rho}(\lambda)V^{\rho}(0)=A^{\nu}{}_{\mu}(\lambda)\Pi^{\mu}{}_{\rho}(\lambda)V^{\rho}(0)\quad\Rightarrow\quad\frac{d}{d\lambda}\Pi^{\nu}{}_{\rho}(\lambda)=A^{\nu}{}_{\mu}(\lambda)\Pi^{\mu}{}_{\rho}(\lambda)$$

Holonomy: formal approach

S Carroll. Spacetime and geometry (2003)

- This differential equation formally integrates as $\Pi^{\mu}{}_{\rho}(\lambda)=\delta^{\mu}_{\rho}+\int_0^{\lambda}A^{\mu}{}_{\sigma}(\eta)\Pi^{\sigma}{}_{\rho}(\eta)d\eta$

  Similarly to what is done when establishing Dyson's formula (QFT), one iterates the process:

$$\Pi^{\mu}{}_{\rho}(\lambda)=\delta^{\mu}_{\rho}+\int_0^{\lambda}A^{\mu}{}_{\rho}(\eta)d\eta+\int_{\eta_2=0}^{\lambda}\int_{\eta_1=0}^{\eta_2}A^{\mu}{}_{\sigma}(\eta_2)A^{\sigma}{}_{\rho}(\eta_1)d\eta_1 d\eta_2+\ldots$$

$$+\int_{\eta_3=0}^{\lambda}\int_{\eta_2=0}^{\eta_3}\int_{\eta_1=0}^{\eta_2}A^{\mu}{}_{\sigma}(\eta_3)A^{\sigma}{}_{\nu}(\eta_2)A^{\nu}{}_{\rho}(\eta_2)d\eta_1 d\eta_2 d\eta_3+\ldots$$

- How to simplify this unpleasant formula ?

  ⇨ Instead of integrating over *n*-simplices, one integrates over *n*-cubes while keeping the product in the right order

$$\int_{\eta_n=0}^{\lambda}\int_{\eta_{n-1}=0}^{\eta_n}\ldots\int_{\eta_1=0}^{\eta_2}A(\eta_n)A(\eta_{n-1})\ldots A(\eta_1)d^n\eta=\frac{1}{n!}\int_{\eta_n=0}^{\lambda}\int_{\eta_{n-1}=0}^{\lambda}\ldots\int_{\eta_1=0}^{\lambda}P\left[A(\eta_n)A(\eta_{n-1})\ldots A(\eta_1)\right]d^n\eta$$

Holonomy: formal approach

S Carroll. Spacetime and geometry (2003)

- Thanks to Taylor's expansion series, the previous formula « miraculously » simplifies into

$$\Pi(\lambda) = I + \sum_{n=1}^{+\infty} \frac{1}{n!} \int_{\eta_n=0}^{\lambda} \int_{\eta_{n-1}=0}^{\lambda} \dots \int_{\eta_1=0}^{\lambda} P\left[A(\eta_n)A(\eta_{n-1})\dots A(\eta_1)\right] d^n\eta = P\exp\int_0^{\lambda} A(\eta)\,d\eta$$

with P is the ordering operator. On a loop γ about a point M, the holonomy writes explicitly as

$$\Pi^{\mu}{}_{\nu}[\gamma] = P\exp\left(-\oint_{\gamma(M)} \Gamma^{\mu}_{\sigma\nu} \frac{dx^{\sigma}}{d\eta} d\eta\right) \Leftrightarrow \Pi[\gamma] = P\exp\left(-\oint_{\gamma(M)} \Gamma_{\sigma} dx^{\sigma}\right)$$

- **Ambrose-Singer theorem** = to know the holonomy at every point of the manifold is equivalent to know the curvature at every point of the manifold ↔ quantum loop gravity.

Holonomy due to a disclination

AM de Carvalho, C Satiro, F Moraes. EPL 80 (2007)

- For a loop about the origin in a z=Cst plane, only the polar connection symbol is retained

$$\Pi[\gamma] = P\exp\left(-\oint_{\gamma(M)} \Gamma_\theta d\theta\right)$$

$$\Gamma_\theta = \frac{m}{\alpha}\left(\alpha^2\cos^2\left[(m-1)\theta+\psi_0\right]+\sin^2\left[(m-1)\theta+\psi_0\right]\right)\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

- For $\underset{(m=1,\ \psi_0=0)}{ds_{2D}^2 = d\rho^2+\alpha^2\rho^2 d\theta^2}$, one gets $\Gamma_\theta = \alpha\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and a Taylor expansion of the parallel propagator gives:

$$\Pi[\gamma] = \begin{bmatrix} \cos(2\pi\alpha) & -\sin(2\pi\alpha) \\ \sin(2\pi\alpha) & \cos(2\pi\alpha) \end{bmatrix}$$

Disclination ⇔ Active rotation = *when acting on a vector, causes its counterclockwise rotation.*

Holonomy due to a disclination

AM de Carvalho, C Satiro, F Moraes. EPL 80 (2007)

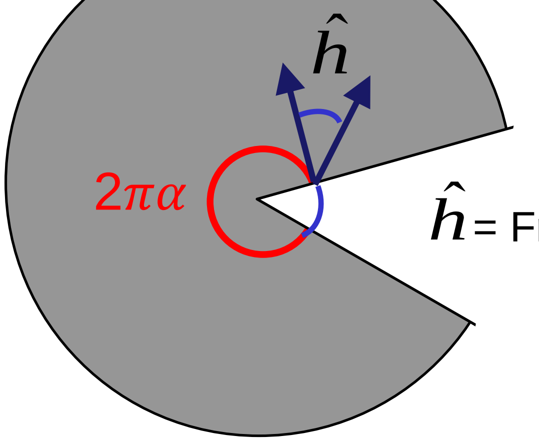
▶ **How much does a vector turn when describing a loop around the defect ?**

$$\hat{h} = 2\pi - 2\pi\alpha$$

Mismatch angle of a vector after a loop

Loop term
θ Î[0,2p]

Active rotation (counterclockwise)



$\hat{h}$ = Frank angle

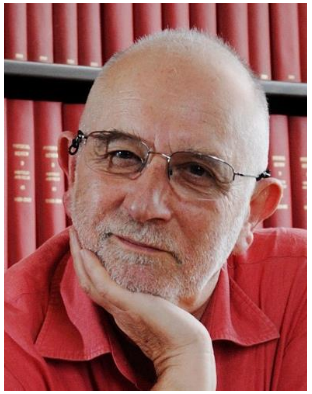
- Gauss-Bonnet version (much more efficient than using Π):

$$\hat{h} = 2\pi\chi - \oint_{\partial\Sigma} \kappa_g ds = 2\pi - \int_{\theta=0}^{2\pi} \frac{1}{R} R\alpha d\theta = 2\pi(1-\alpha)$$

$$ds^2 = \alpha^2 R^2 d\theta^2$$ *circle of radius R*

Let us do the experiment !!

Comparison with Aharonov-Bohm

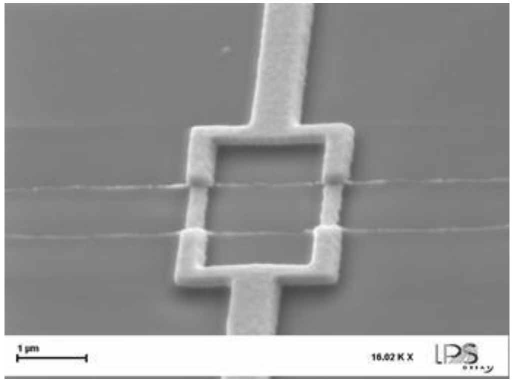


M Berry

« The Aharonov-Bohm effect is real physics not ideal physics ».

Aharonov-Bohm phase	***Moraes phase***
Magnetic flux is confined within the solenoid, it vanishes everywhere else.	Curvature is confined within the disclination line, it vanishes everywhere else.
It has measurable effects: shift of the electronic interference pattern…	It has measurable effects: rotation of polarization plane of linearly polarized light…
Quantum	Classical

SQUID



Still waiting for its NDC device…

A tentative definition

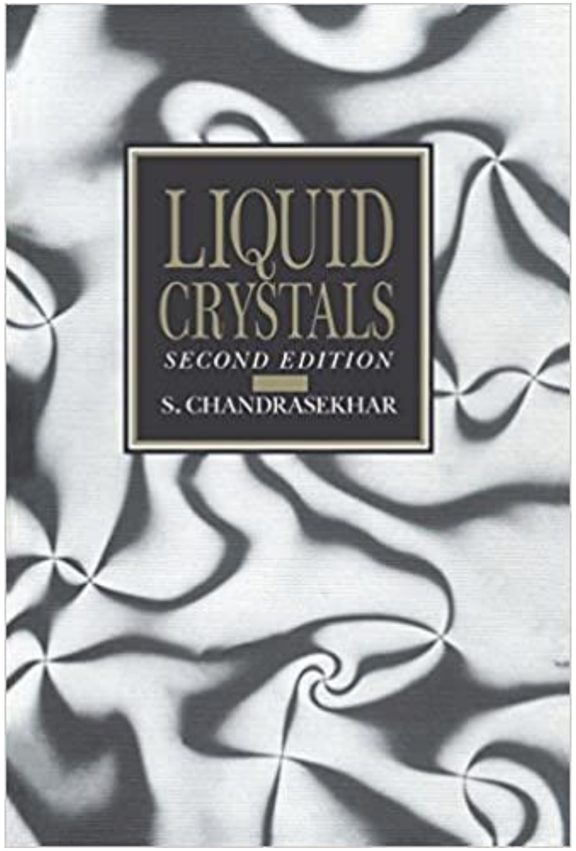
E Cohen et al. Nature Reviews 1 (2019)

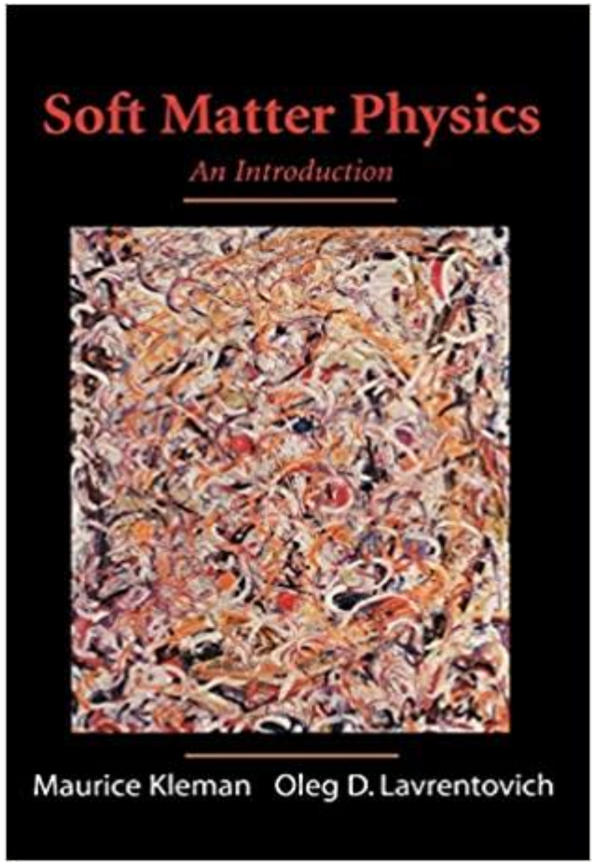
Phase	First appeared in	Mostly known in	Parameter space	Topological	Adiabatic
Pancharatnam	1956	Optics	Poincaré sphere	No	Yes
Aharonov-Bohm	1959	Quantum electrodynamics	Spacetime	Yes	No
Exchange statistics (of Abelian anyons)	1977 1982 1984	Condensed matter	Real space	Yes	Yes
Berry	1983 1984	Quantum mechanics	General	No	Yes
Aharonov-Casher	1984	Quantum electrodynamics	Real space	Yes	No
Hannay angle	1985	Classical mechanics	Real space	No	Yes
Aharonov-Anandan	1987	Quantum mechanics	General	Yes	No
Zak	1989	Condensed matter	Momentum space	No	No

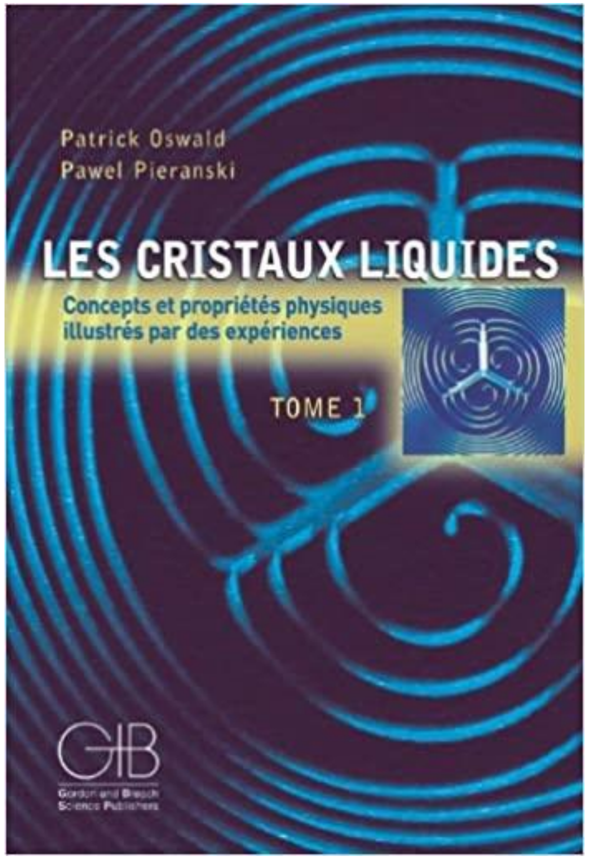
They are all examples of what is generically called *geometric* or *Berry phases*, that is « **phases are not attributed to the forces applied onto the [quantum] system. Instead, they are associated with the connection of space itself.** »

Next lecture…

Bibliography

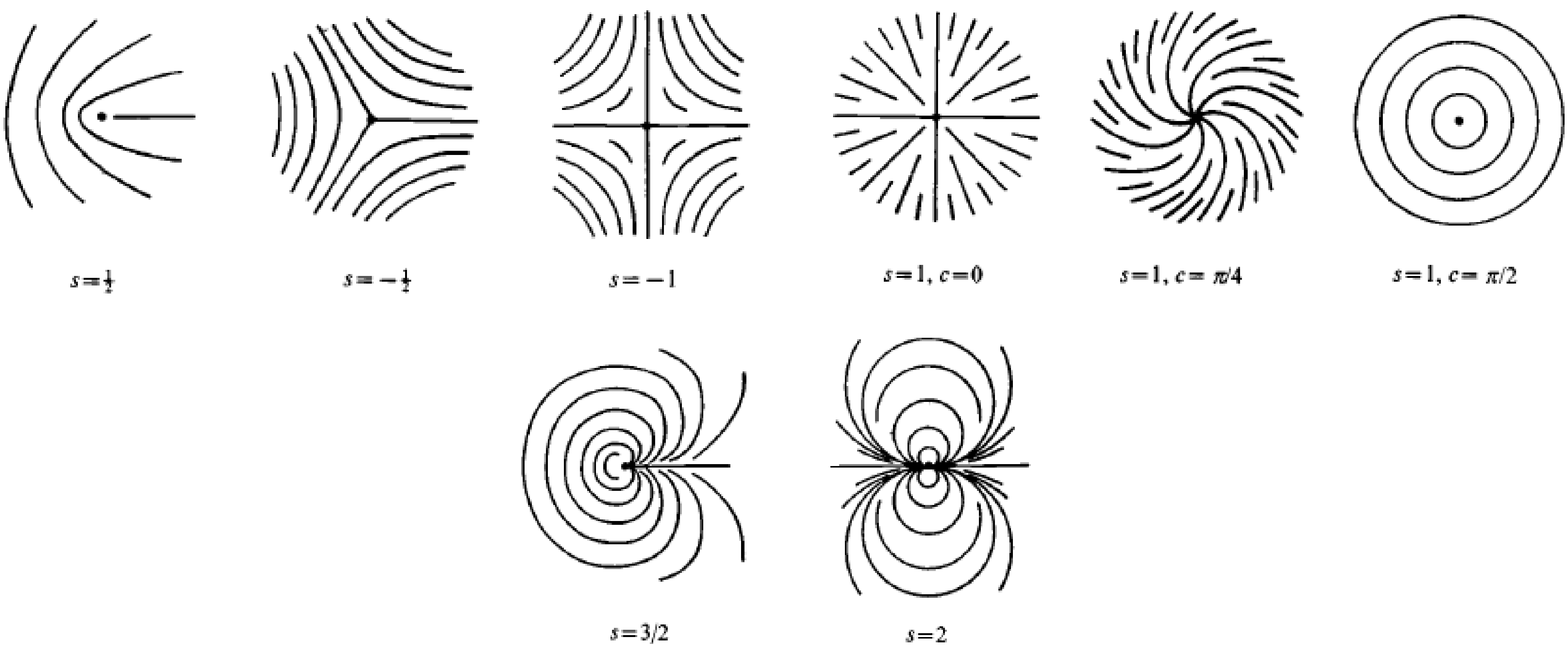




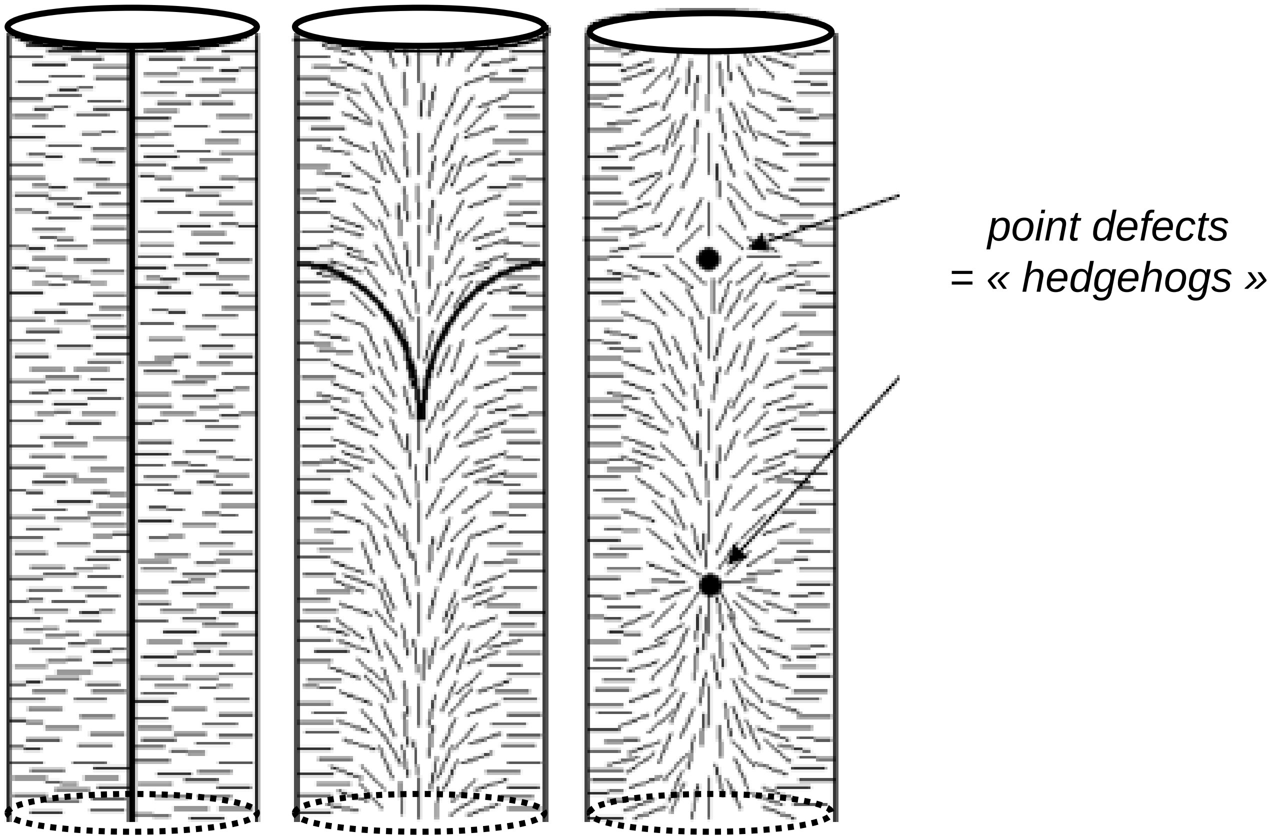


Appendices

Disclination gallery



Escape in the third dimension



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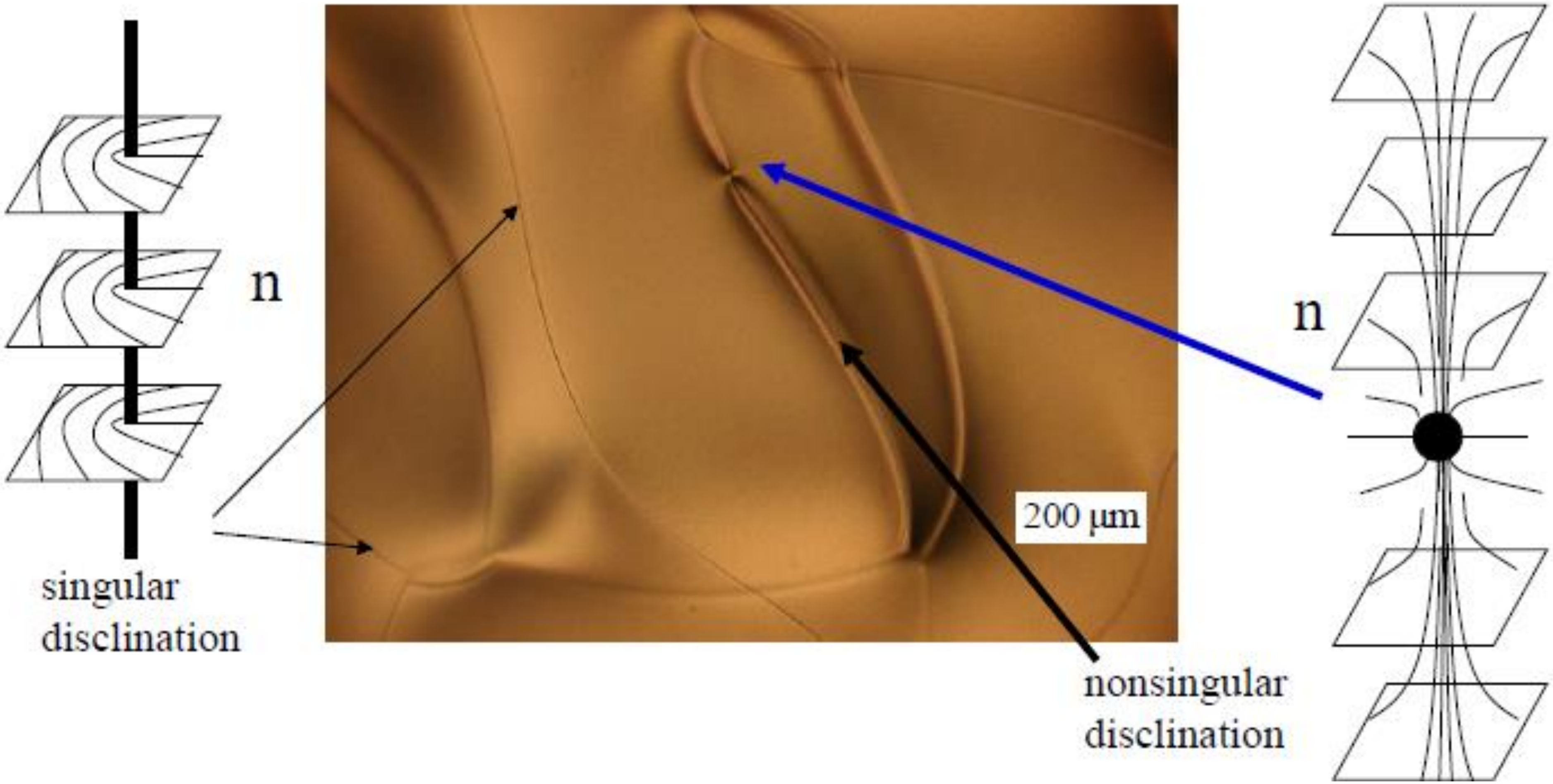
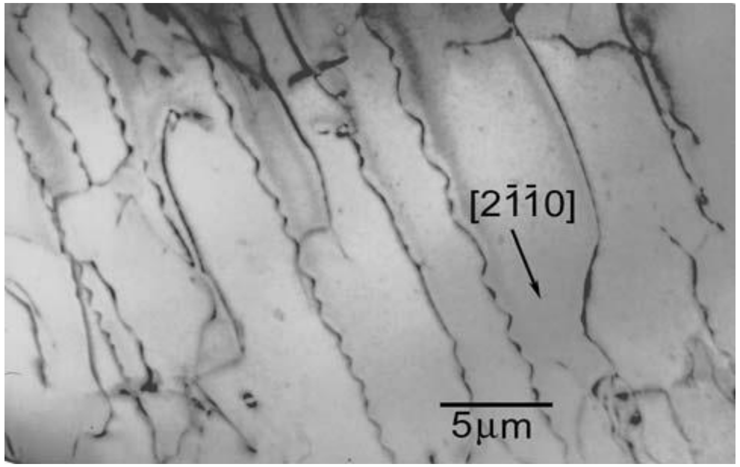


O Lavrentovich. Kent state University

Twisting things around

- Line defects can also have a chirality. For instance in crystalline solids, a very common chiral defect is the screw dislocation:



TEM image in $CaCO_3$

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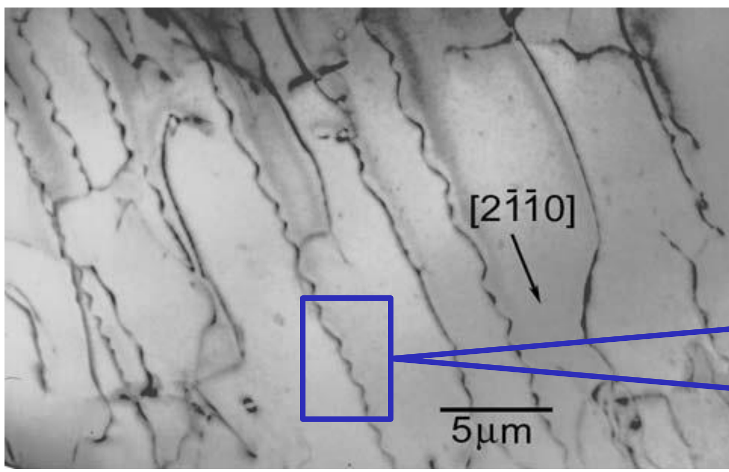
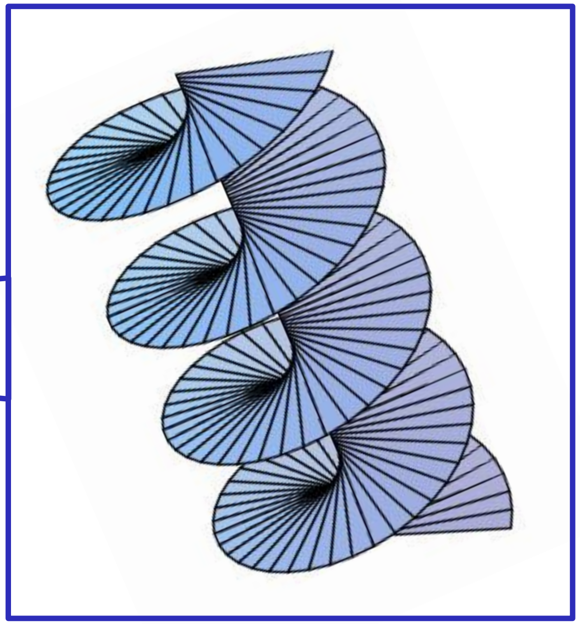


TEM image in $CaCO_3$



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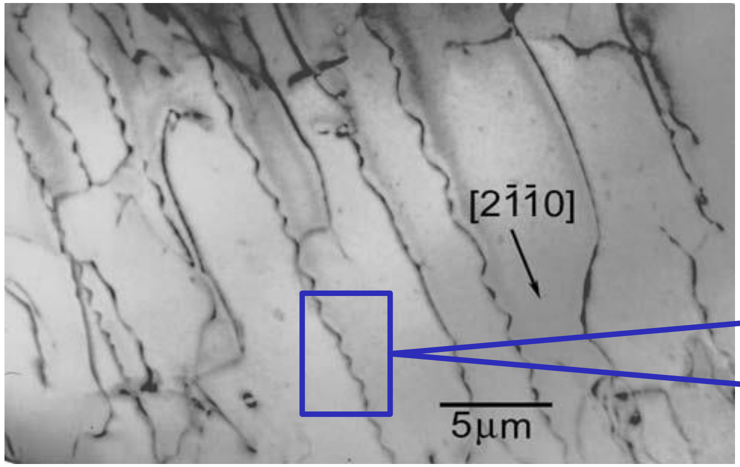

TEM image in $CaCO_3$

A « giant screw dislocation » in Da Vinci's spiral stairway (Chambord)

Twisting things around

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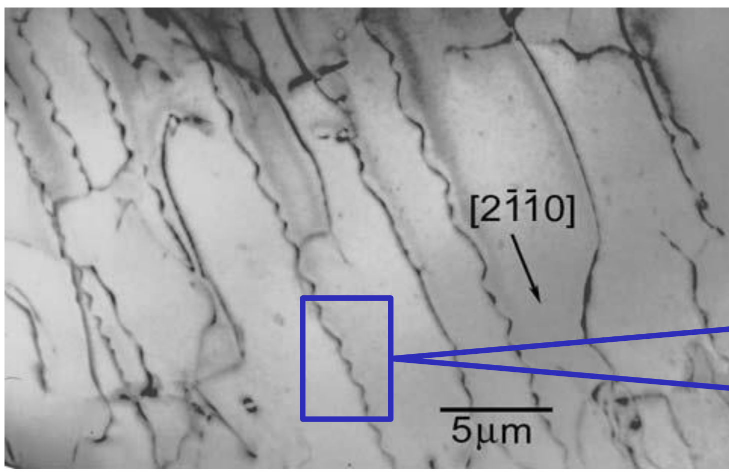
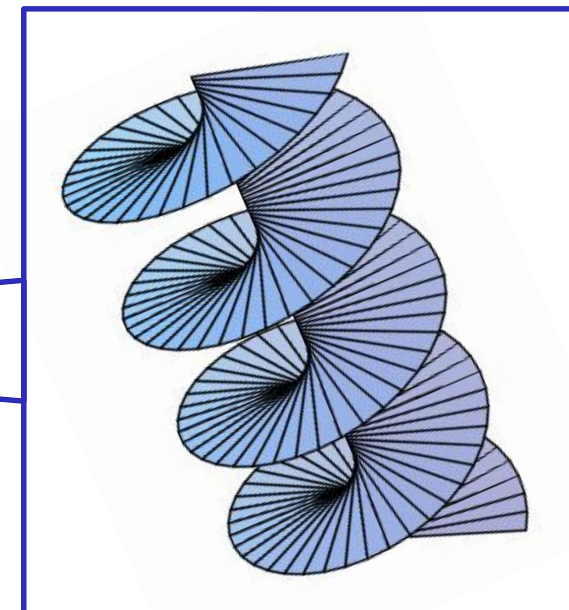


TEM image in $CaCO_3$

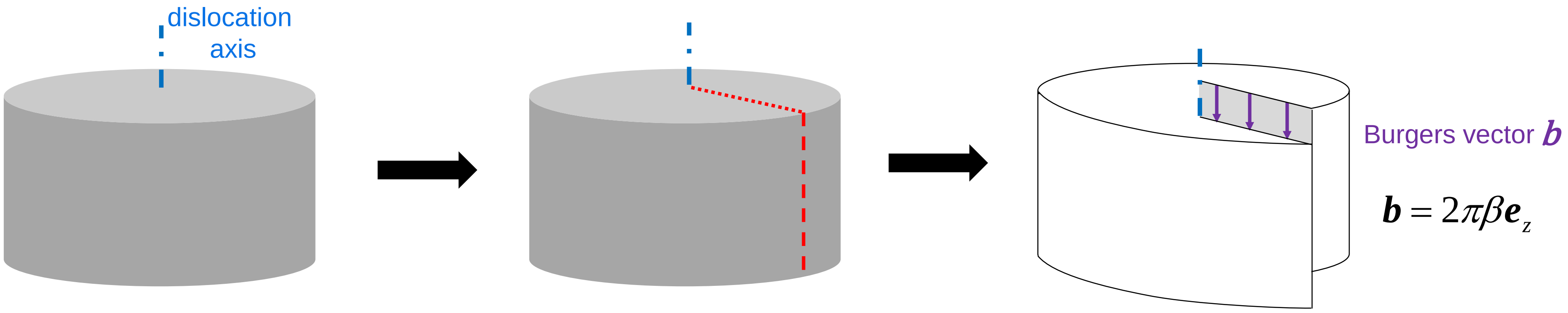


- The defect couples the rotation and translation around the z-axis: $\theta \to \theta + 2\pi \Leftrightarrow z \to z + 2\pi\beta$

  The sign of β dictates if the helix is left-handed of right-handed.

Twisting things around

- Volterra process for the screw dislocation:



$$\boldsymbol{b} = 2\pi\beta\boldsymbol{e}_z$$

Twisting things around

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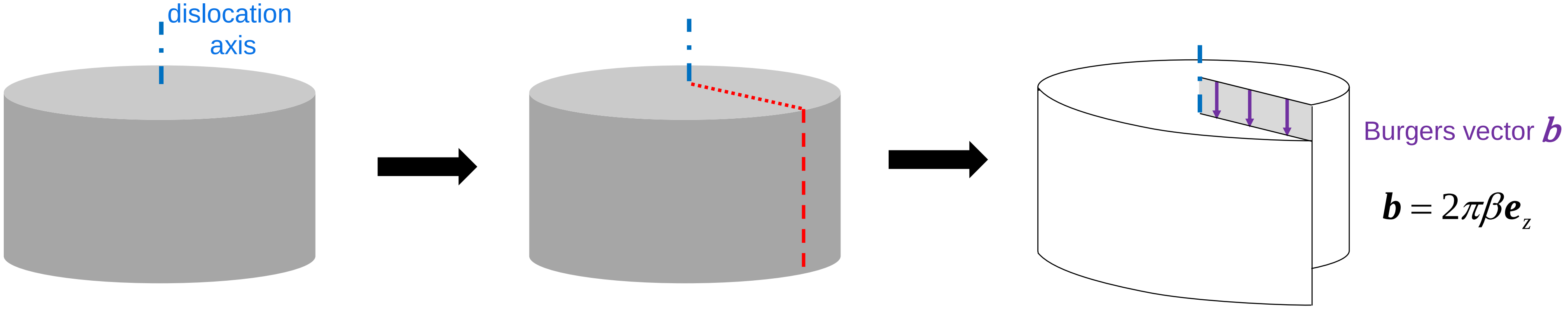
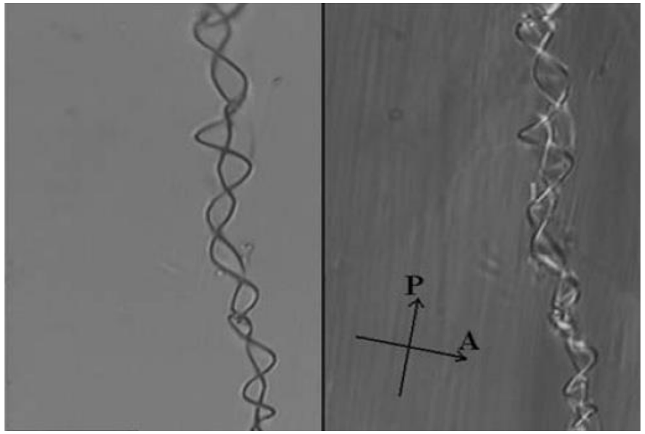


- In mesophases as well, chiral line defects have been observed:



POM image in a smectic (C. Blanc, Montpellier 2)

Twisting things around

- The geometry of a single twisted disclination (or dispiration) is given by the line element:

$$ds_{3D}^2 = dr^2 + \alpha^2 r^2 d\theta^2 + \left(\beta d\theta + dz\right)^2$$

- Differential geometry of a dispiration is much harder than it may look at first sight:

1. *Distributional non-vanishing curvature on the z-axis:*

$$R(r) = \frac{(1-\alpha)}{\alpha r}\delta(r)$$

V_4 = Riemann geometry

2. *Distributional non-vanishing torsion on the z-axis:*

$$T^z = 2\pi\beta \frac{\delta(r)}{r} dr \wedge d\theta$$

U_4 = Riemann-Cartan geometry

Distribution of defects

S Fumeron et al. Eur. Phys. J. B 90 (2017)
MO Katanaev et al.. Annals of Physics (1999)

- In real-life, defects are not isolated. Sometimes, it is possible to find analytical expressions for the line element:

1. *Discrete distribution of disclinations:* $ds^2 = -c^2 dt^2 + e^{-4V(x,y)}\left(dx^2 + dy^2\right) + dz^2$

$$V(x,y) = \frac{|F|}{4\pi} \ln\left[\left(\frac{\cosh^2\left(\frac{\pi}{2a}(y-b)\right) - \cos^2\left(\frac{\pi x}{2a}\right)}{\cosh^2\left(\frac{\pi}{2a}(y-b)\right) - \sin^2\left(\frac{\pi x}{2a}\right)}\right)\left(\frac{\cosh^2\left(\frac{\pi}{2a}(y+b)\right) - \sin^2\left(\frac{\pi x}{2a}\right)}{\cosh^2\left(\frac{\pi}{2a}(y+b)\right) - \cos^2\left(\frac{\pi x}{2a}\right)}\right)\right]$$

2. *Continuous distribution of dislocations:* $ds_{3D}^2 = dr^2 + r^2 d\theta^2 + \left(\Omega r^2 d\theta + dz\right)^2$

$\Omega = Nb/2$ *Surface density of Burgers vector*

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3. *Continuous distribution of chiral disclinations (r>R):*

$$ds_{3D}^2 = dr^2 + r^{2(A-1)} d\theta^2 + \left(\Omega r^2 d\theta + dz\right)^2$$

$A = 1 + qR^2/2$ *Surface density of deficit angle*