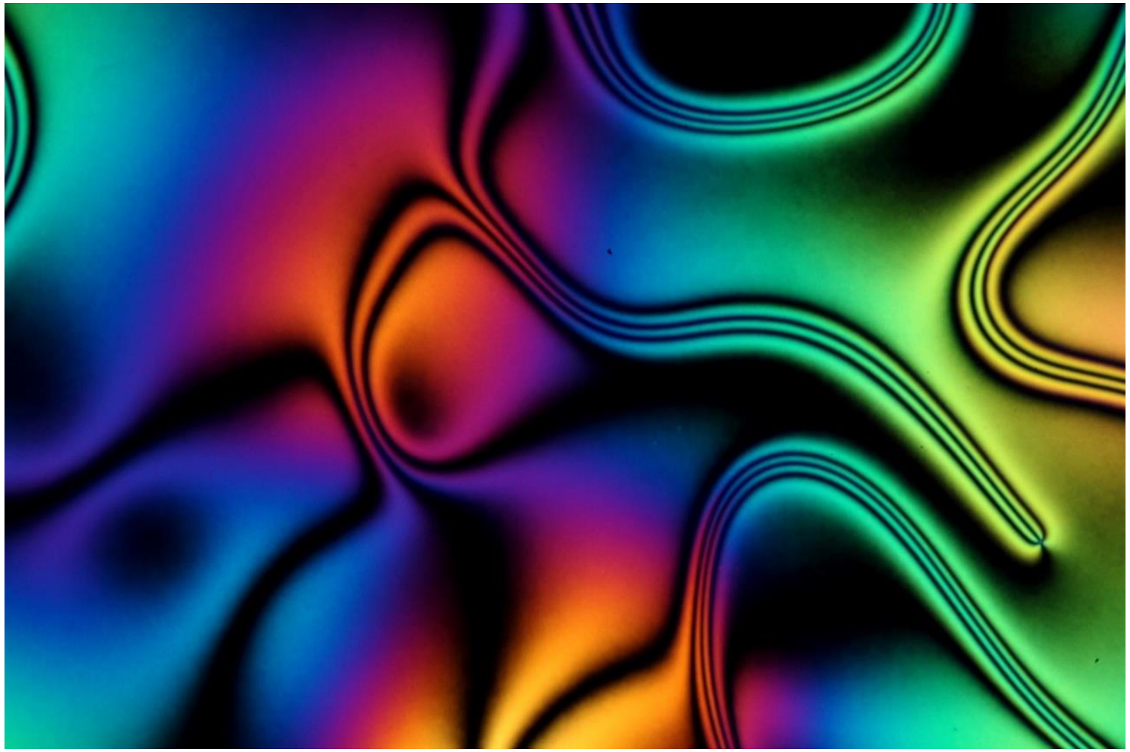
Sébastien Fumeron

Thematic School

New trends in condensed matter

Introduction to topological defects

Liquid crystals, active matter and more…



UNIVERSITÉ DE LORRAINE

Laboratoire Physique et Chimie Théoriques

Condensed matter

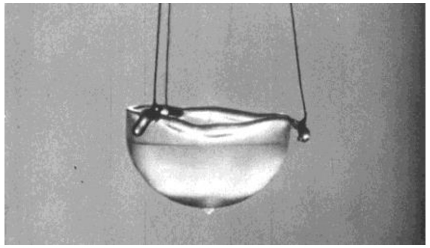
Crystalline solids

Isotropic fluids

Insulators

Liquids

Metals and superconductors



Superfluids

They are rigid

They can flow

Condensed matter

Crystalline solids

Isotropic fluids

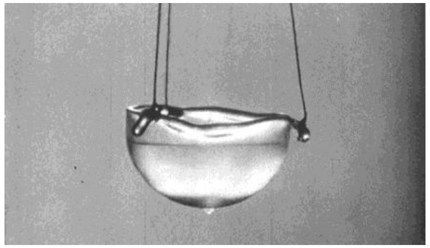
Insulators

Sand ?

Liquids

Metals and superconductors

Gel (agar-agar) ?



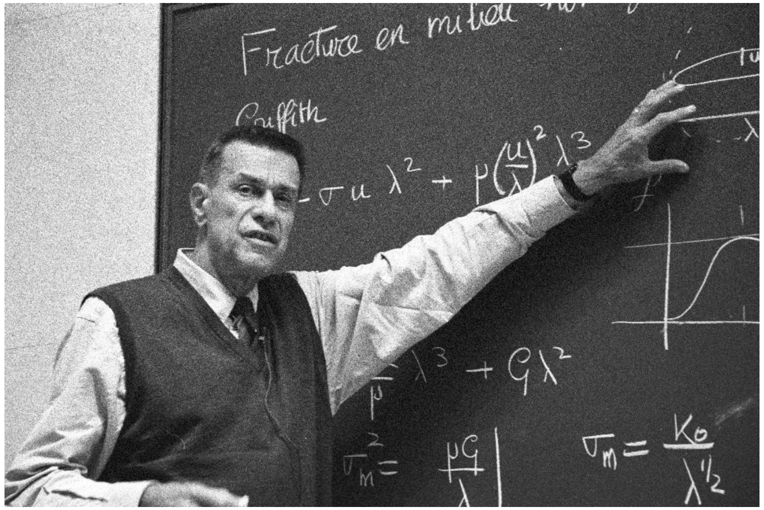
Superfluids

They are rigid

They can flow

« Soft matter »

The term was coined by Pierre-Gilles de Gennes in the title of his 1991 Nobel Prize conference. Soft condensed matter is defined as:



PG De Gennes

- media with « *large response functions* » (PG De Gennes).
- « *materials that will not hurt your hand if you hit them* » (T Lubensky) = viscoelastic materials, i.e. media that exhibit properties of both solids (they deform elastically) and liquids (they flow).
- « *an amalgamation of methods and concepts* » *from « physics, chemistry, engineering, biology, materials, and mathematics departments. The problems that soft matter…examines are the interdisciplinary offspring that emerge from these otherwise distinct fields »* (J Silverberg).

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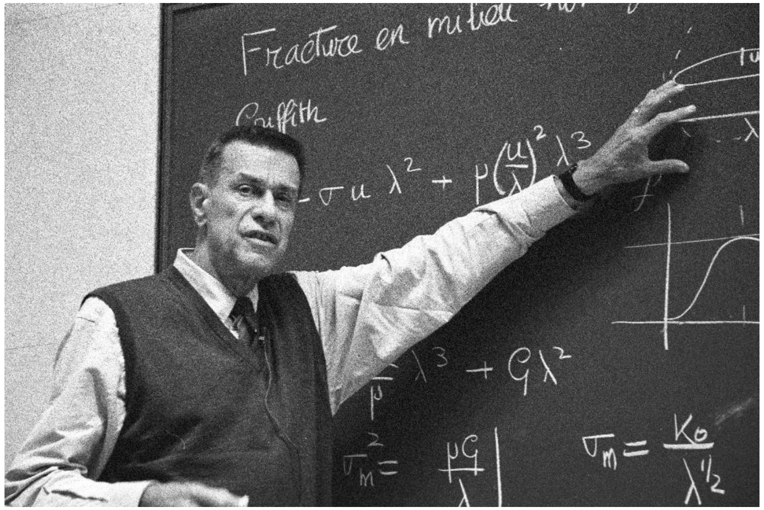

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Examples: colloids (emulsions, foams, paints, aerosols, gels and glues), granular media, **liquid crystals**…

Many open problems: active matter, glass transition, flows of granular materials, sonoluminescence…

Overview of the lectures

▸ Part 1: A crash course on liquid crystals

- The nematic mesophase
- Topology and phase transitions

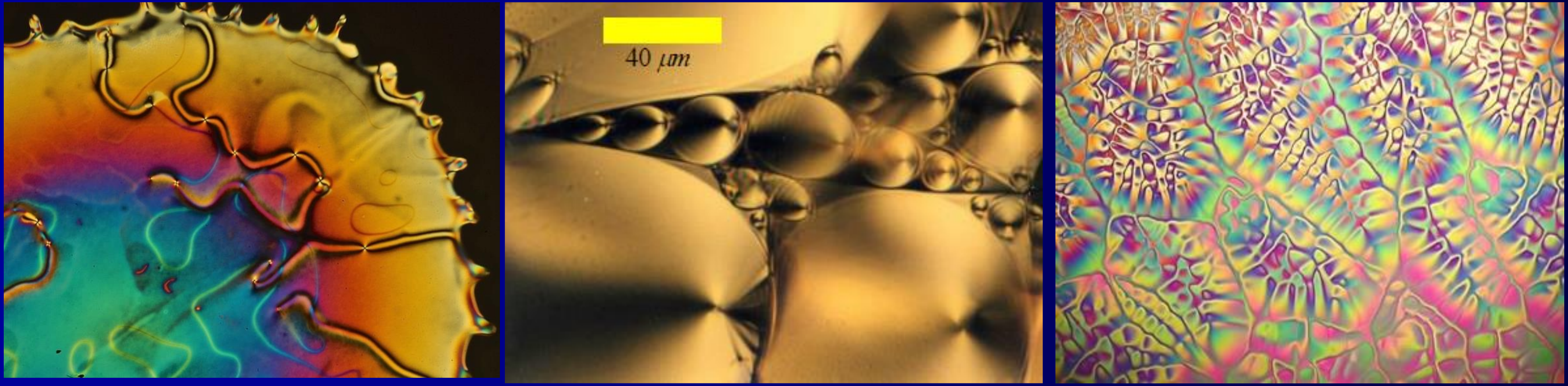
▸ Part 2: Topological defects in nematics

- Half and whole-integer disclinations
- Differential geometry
- Geometric phases

▸ Part 3: Topological defects everywhere ?

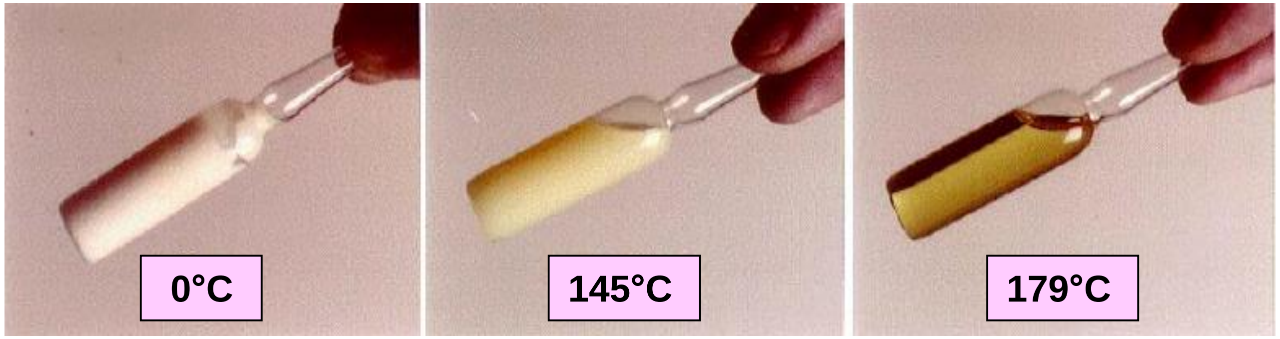
- Biology
- Cosmology

Part I. A crash course on liquid crystals



A story of carrot and stick

F Reinitzer



In 1888, botanist Friedrich Reinitzer studied crystals of cholesteryl benzoate from carrots and observed that they have two melting points. When reaching T=145°C, crystals melt and turn into a milky fluid. When reaching T=179°C, it changes again into a clear liquid.

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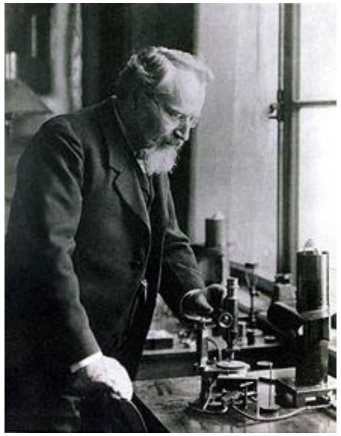

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Reinitzer asked physicist Otto Lehmann about his observations. With the help of a polarizing microscope, Lehmann noted in 1890 that the cloudy phase flowed like a liquid, but also had a stick-like molecular structure that was somewhat ordered as in solid crystals. He coined the term “liquid crystals”.

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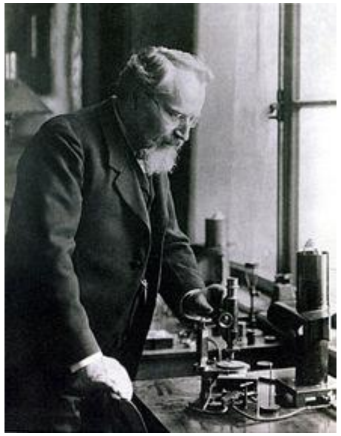

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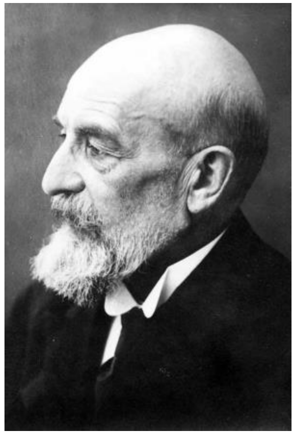

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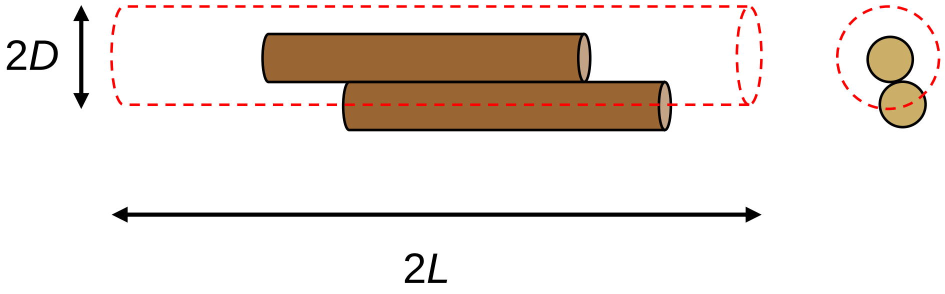
In 1922, Georges Friedel realized the milky fluid is a new phase of matter, in-between an isotropic liquid and a crystalline solid: a **mesophase**. He classified mesophases into nematic, smectic and cholesteric phases. Today, there are columnar phases, blue phases…

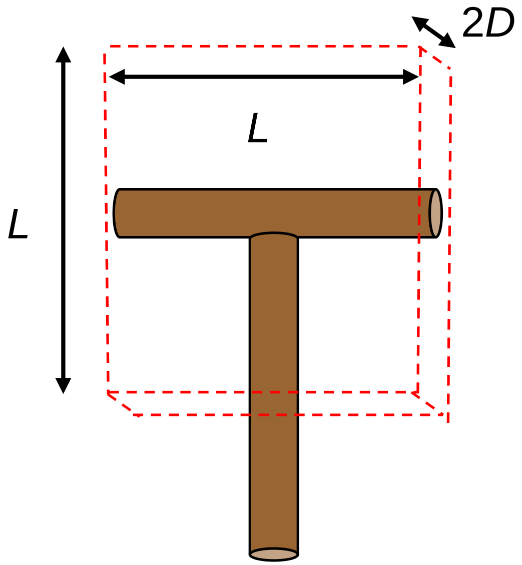
Onsager-Flory point of view

- **How can an assembly of rodlike molecules arrange itself in a given volume ?**
  - Assumptions: 1) *Attractive intermolecular forces are not taken into account.*
    2) *Hard-rod model (diameter D, length L>>D)* $\Rightarrow$ *steric effects*.



minimal excluded volume

$V_m = 2\pi D^2 L$



maximal excluded volume

$V_M = 2DL^2$

Onsager-Flory point of view

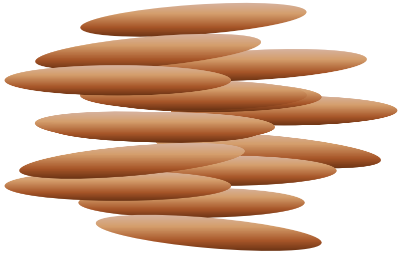
▶ How can an assembly of rodlike molecules arrange itself in a given volume ?

- In order to maximize the configuration entropy $S=k_B \ln \Omega$ (= to increase the free volume available per molecule), the parallel configuration with a minimum excluded volume is favored by the assembly.

⇨ *The matchbox problem*

- Orientational ordering decreases orientational entropy but increases positional entropy. Onsager showed that positional entropy can be gained at the expense of orientational entropy only <u>beyond a certain density</u>.

Mesogenic anatomy



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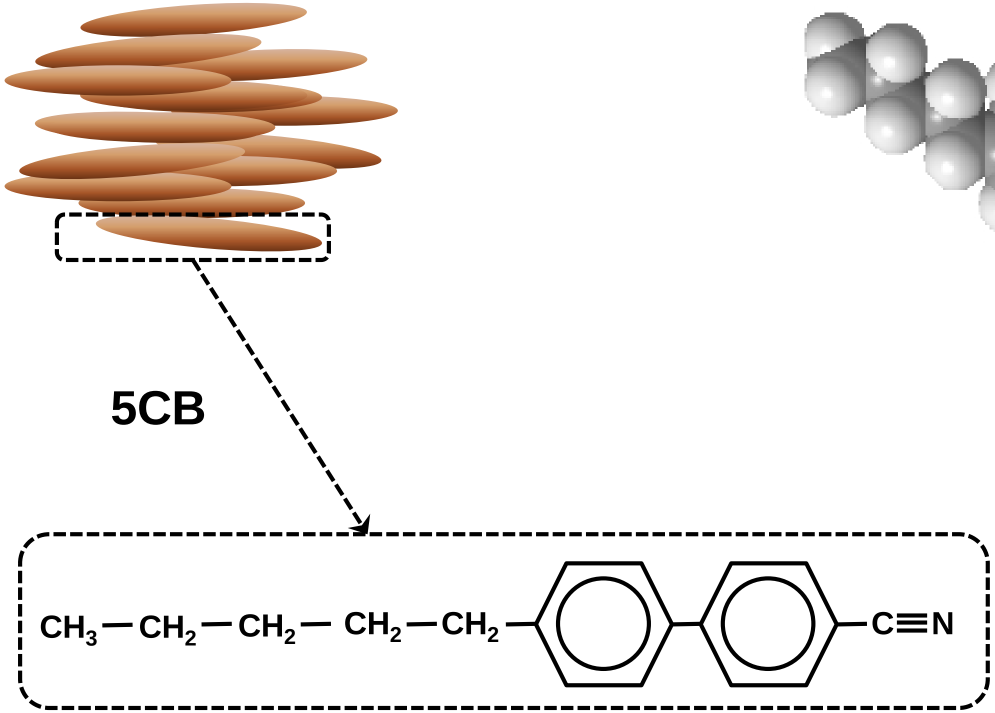

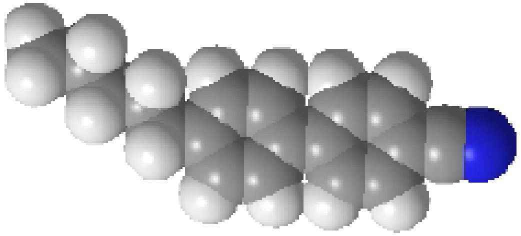

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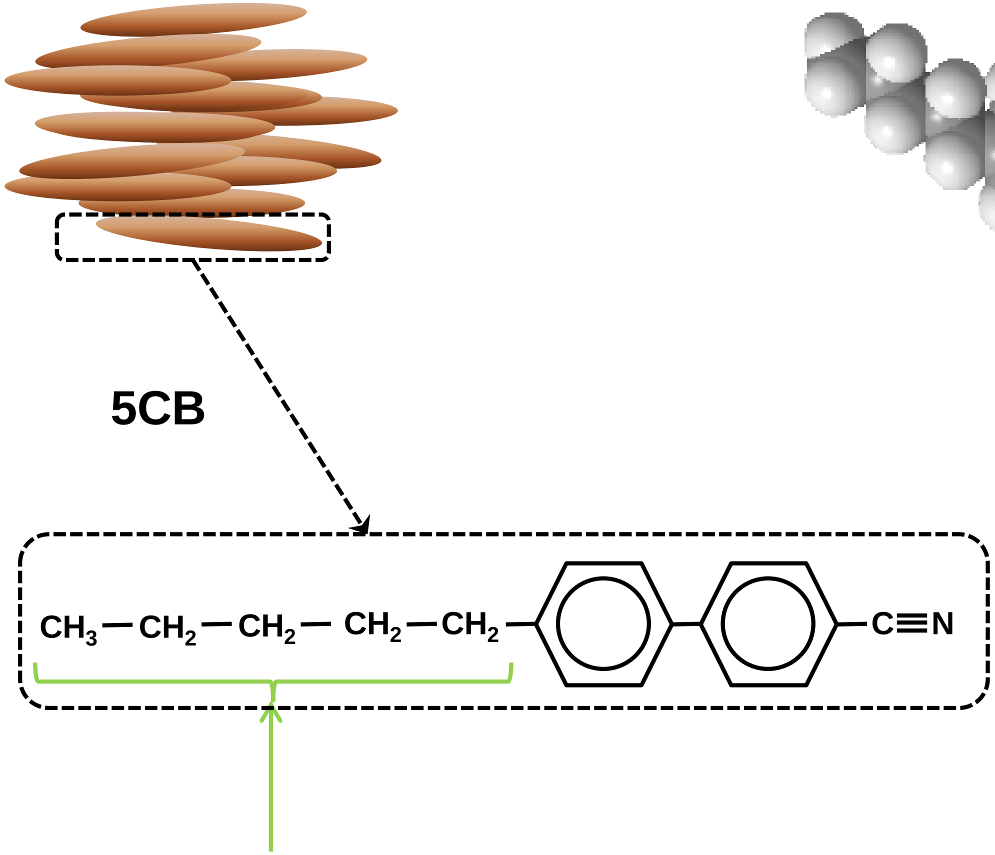
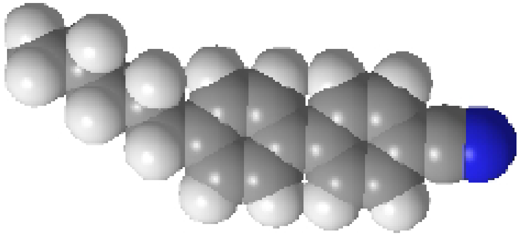
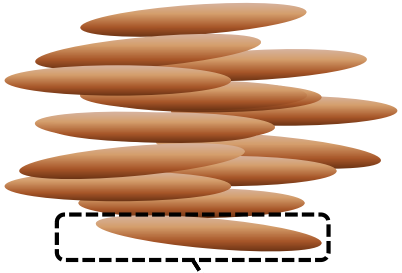
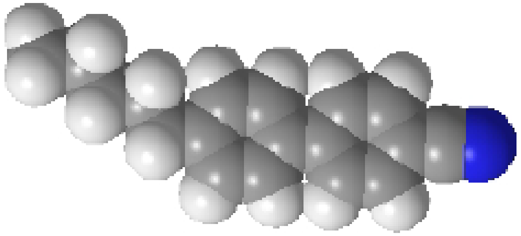




aliphatic chain ⇒ flexible outer part ⇒ **fluidity**

Mesogenic anatomy



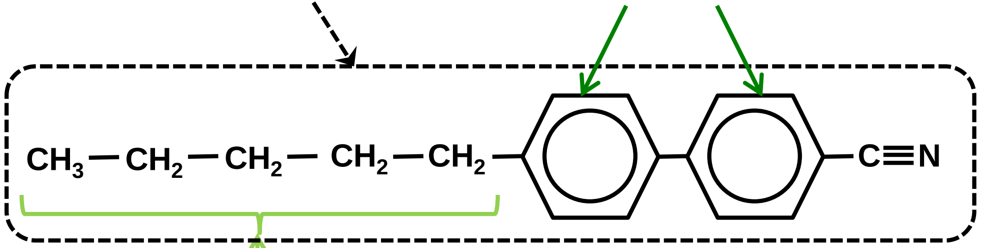


rigid core

5CB

phenyl groups ⇒

- orientational order ⇒ **anisotropy**
- π-electron cloud ⇒ **high polarizability**



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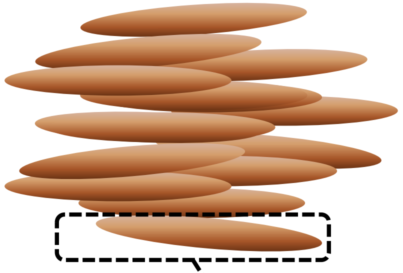

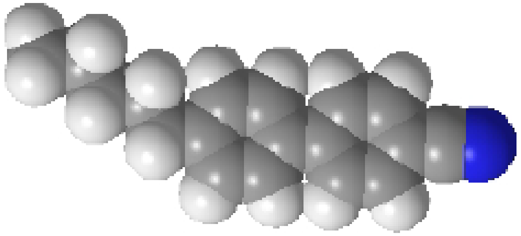

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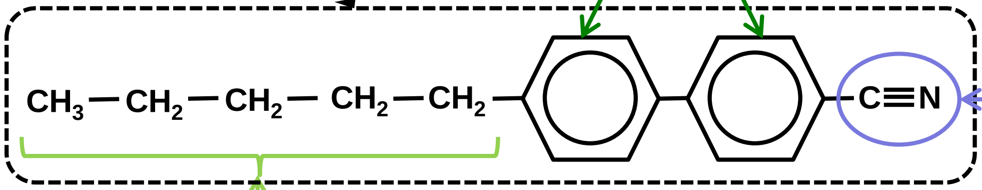
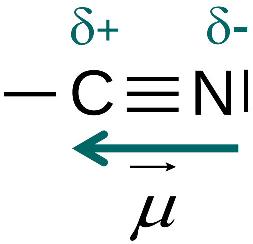


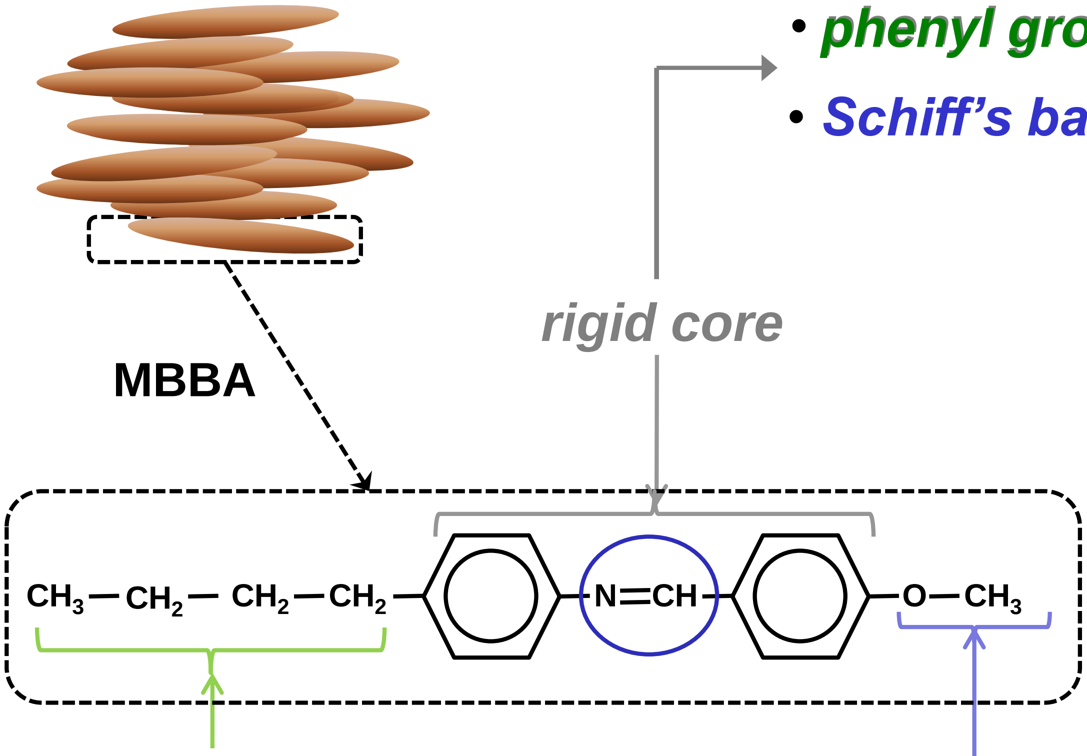
carbonitrile group ⇒ strong longitudinal **dipole moment**

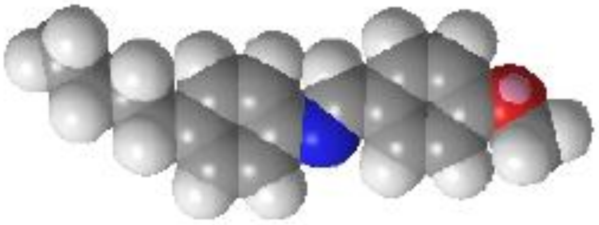


aliphatic chain ⇒ flexible outer part ⇒ **fluidity**

Mesogenic anatomy

- ***phenyl groups*** ⇒ **anisotropy + polarisability**
- ***Schiff's base*** ⇒ strong **dipole moment**





aliphatic chains

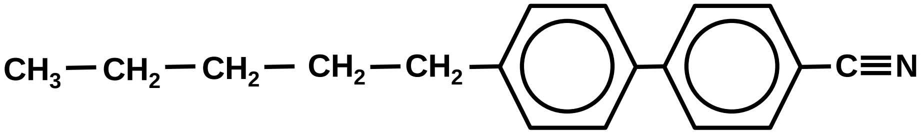
⇒ flexible outer parts ⇒ **fluidity**

methoxy group ⇒ transverse **dipole moment**

Maier-Saupe-like standpoint

▶ **Orientation-dependent interaction between the molecules**:

- The nematic phase is a compromise between the attractive Van der Waals interactions that align rigid cores on average along the same direction (anisotropy) and the thermal agitation of the aliphatic chains increasing the mean steric hindrance (fluidity).



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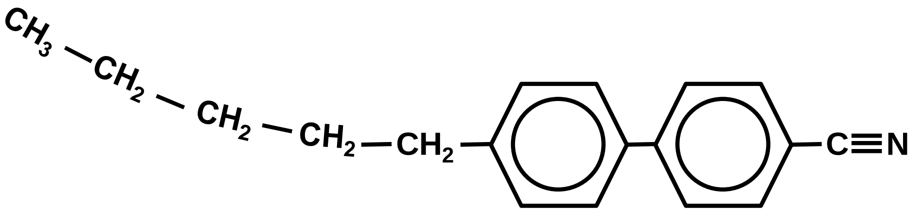

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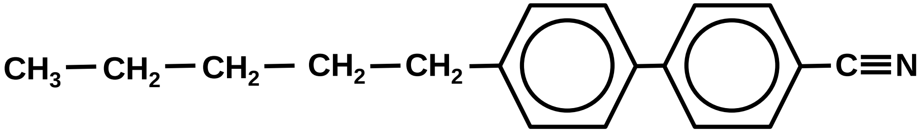

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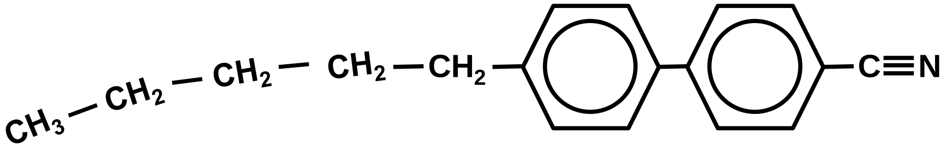

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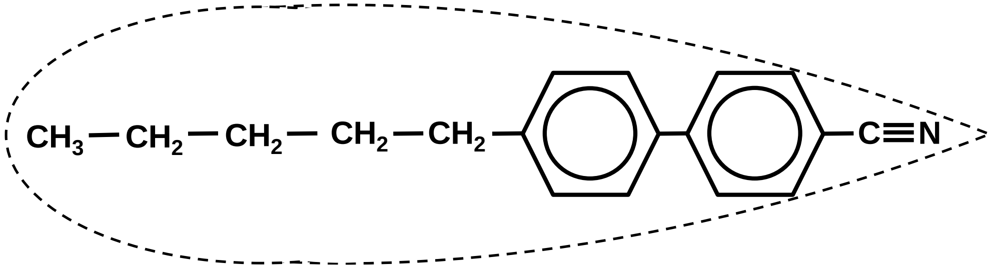

Maier-Saupe-like standpoint

Leadbetter et al. J. Physique **C1 26** (1975)

▸ Orientation-dependent interaction between the molecules:

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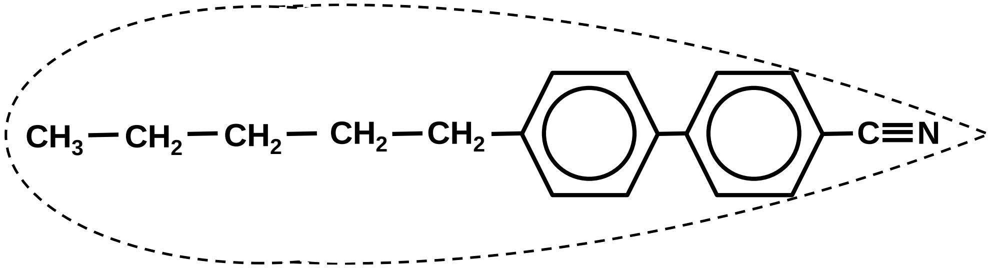
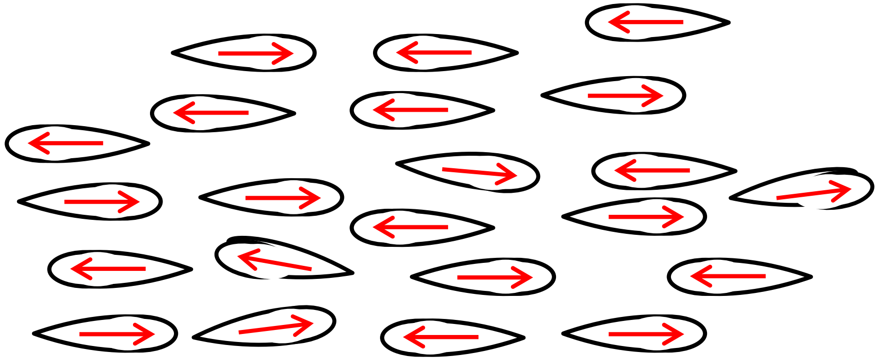




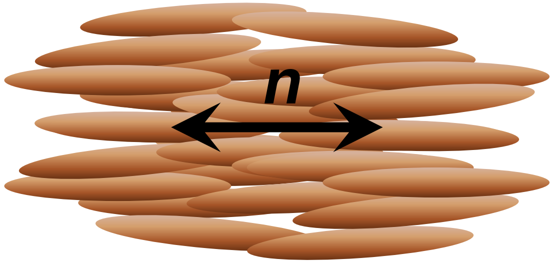
- Dipole-dipole align anticolinearly: dimeric head-tail structure was confirmed by X-ray diffraction in 5CB and 7CB.

⇨ Statistically, the assembly is unchanged when inverting heads and tails:

Z_2 group = symmetry group of the nematic state

The director field

▶ How to characterize the assembly of rodlike molecules ?



- Orders of magnitude:
  - nematogen width ≈ Å
  - nematogen length ≈ nm
  - orientational correlation length ≈ µm

⇒ long-range orientational order (and no positional order)

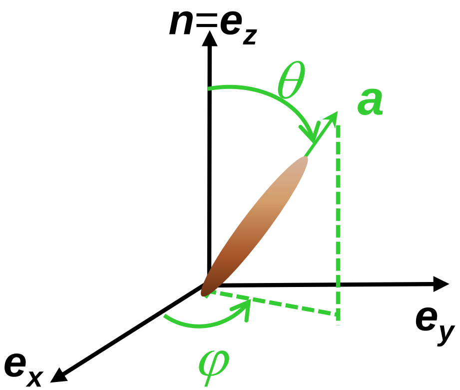
- The local average orientation of nematogens is given by a dimensionless unit “vector” ***n*** called the **director field** = symmetry-axis of the assembly of rods.

⇒ The nematic phase is statistically invariant under the elements of

$SO(2) \times Z_2 = O(2)$

- A particular nematogen will generally not point exactly in the direction of ***n***:

$$\boldsymbol{a} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$



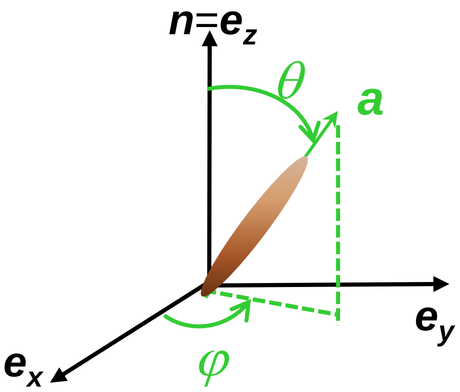
The director field

▶ How to characterize the assembly of rodlike molecules ?

- Let $f\left(\theta,\varphi\right)$ be the probability to find the nematogen pointing in the direction ***a*** (within a solid angle $d\Omega=\sin\theta d\theta d\varphi$), then

1) f must not depend on φ (cylindrical symmetry)

2) $f\left(\theta\right)=f\left(\pi-\theta\right)$ (Z_2 symmetry)



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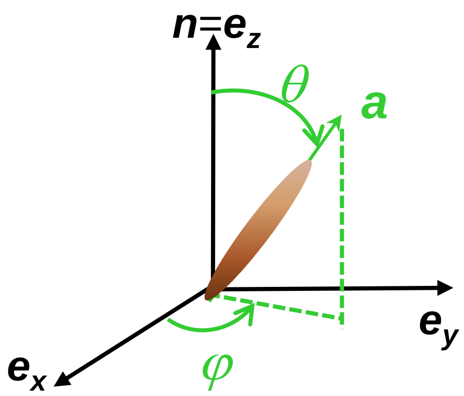
⇨ First guess to characterize orientational order: a dipolar scalar parameter

$$< \cos\theta >= \int_0^\pi f\left(\theta\right)\cos\theta \times 2\pi \sin\theta d\theta$$



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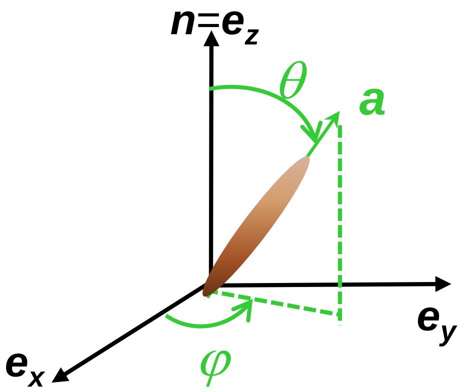
$$< \cos\theta > = \int_0^\pi f\left(\theta\right)\cos\theta \times 2\pi \sin\theta d\theta = 0 \ \forall f \ \ !!$$



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⇨ Second guess: a quadrupolar scalar parameter $< \cos^2\theta >$

Ideally, the parameter should be normalized and mark a clear difference between a narrow distribution about 0(π) and a random one.

The director field

▶ How to characterize the assembly of rodlike molecules ?

- Tsvetkov (1942)

$$\boxed{S = < P_2(\cos\theta) > = \left\langle \frac{3}{2}\cos^2\theta - \frac{1}{2} \right\rangle}$$

- Specific cases:

1) Isotropic orientation $$f(\theta) = \frac{1}{4\pi} \Rightarrow S = 0$$

2) Perfectly aligned rods $$\cos\theta = \pm 1 \Rightarrow S = 1$$

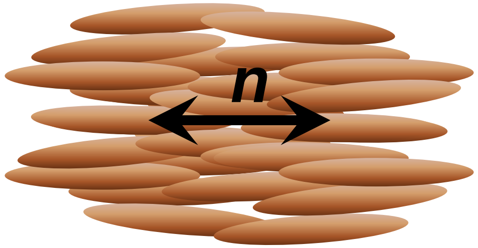
3) Realistic nematic state $$0,3 \leq S \leq 0,8$$

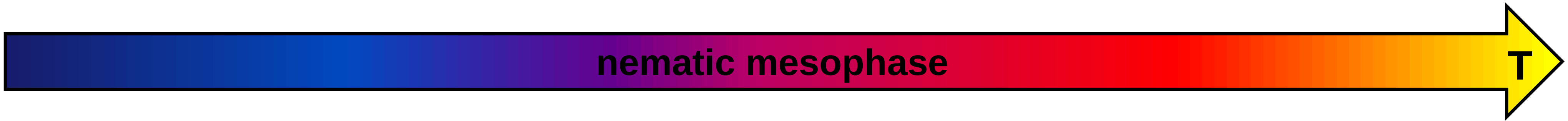
- Can we do better ? Yes, a traceless symmetric second-rank tensor, the order parameter tensor or Landau-De Gennes Q-tensor, defined as

$$Q_{ij} = S\underbrace{\left(\frac{3}{2}n_i n_j - \frac{1}{2}\delta_{ij}\right)}_{\text{what is the direction of the alignment}}$$

how strongly molecules are aligned

What happens when the temperature changes ?





orientational order

no positional order

$0 \leq S \leq 1$

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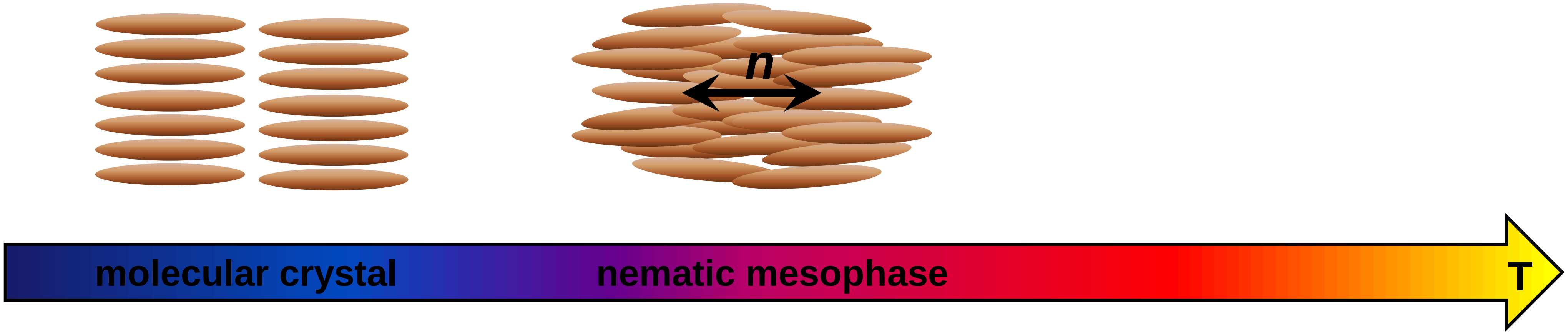
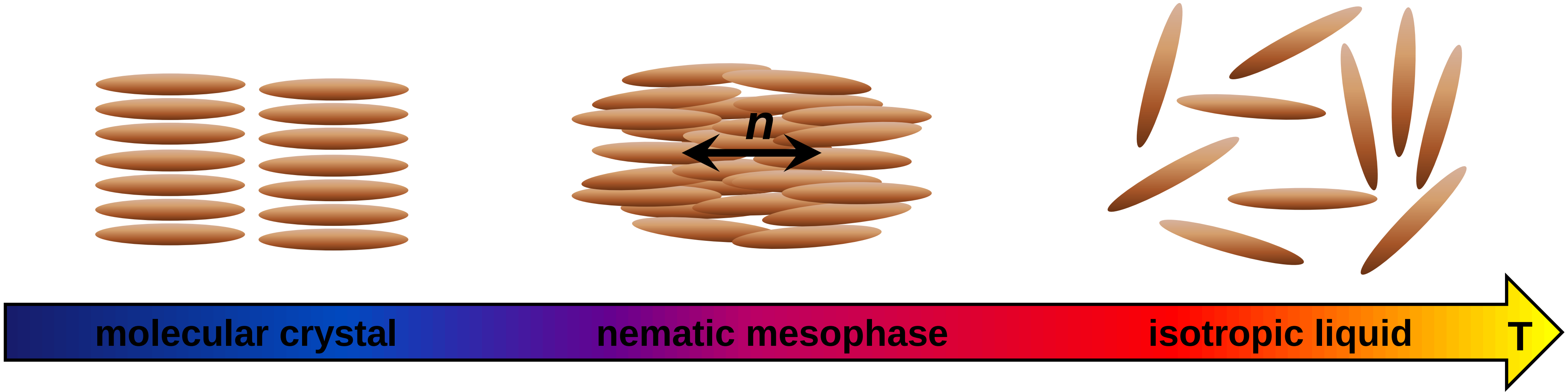


molecular crystal **nematic mesophase**

orientational order ↔	***orientational order***
positional order	*no positional order*
$S \sim 1$	$0 \leq S \leq 1$

- At low temperatures, thermal agitation is weak (low steric hindrance $\Rightarrow$ closer nematogens) and Van der Waals interactions prevail $\Rightarrow$ molecular solid. Sometimes: smectic phases.

What happens when the temperature changes ?

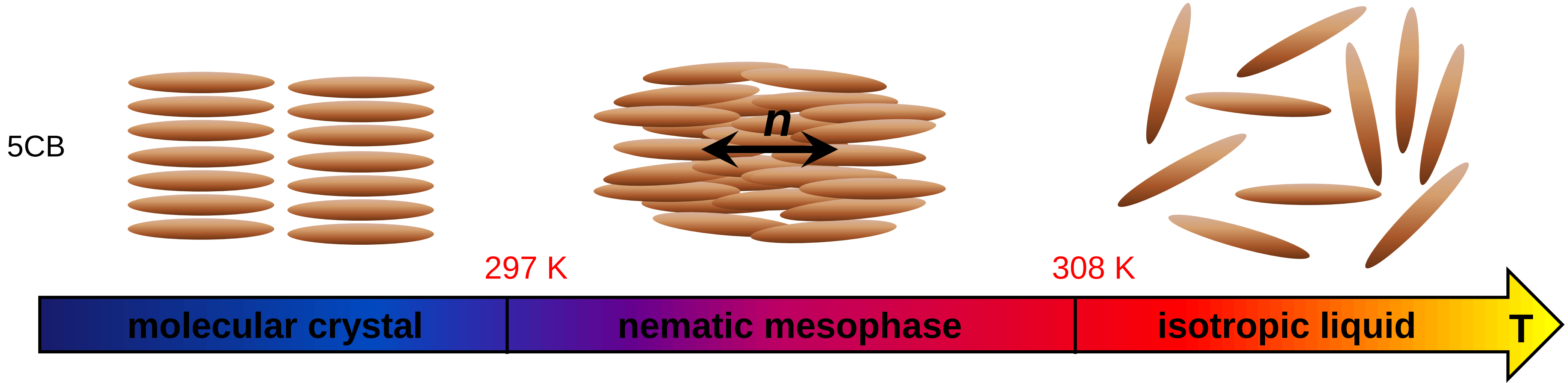


molecular crystal **nematic mesophase** **isotropic liquid**

orientational order	↔	***orientational order***		*no orientational order*
positional order		*no positional order*	↔	*no positional order*
$S \sim 1$		$0 \leq S \leq 1$		$S = 0$

- At low temperatures, thermal agitation is weak (low steric hindrance ⇒ closer nematogens) and Van der Waals interactions prevail ⇒ molecular solid. Sometimes: smectic phases.
- At high temperatures, thermal agitation prevails over Van der Waals interactions (nematogens are distant from each other + force in d^{-6}) ⇒ isotropic fluid. Sometimes: blue phases.

What happens when the temperature changes ?



orientational order ↔ ***orientational order*** SSB *no orientational order*

positional order SSB *no positional order* ↔ *no positional order*

$S \sim 1$ $0 \leq S \leq 1$ $S = 0$

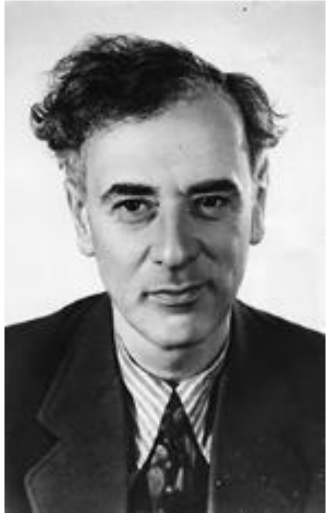
- At low temperatures, thermal agitation is weak (low steric hindrance ⇒ closer nematogens) and Van der Waals interactions prevail ⇒ molecular solid. Sometimes: smectic phases.
- At high temperatures, thermal agitation prevails over Van der Waals interactions (nematogens are distant from each other + force in d^{-6}) ⇒ isotropic fluid. Sometimes: blue phases.

⇒ Between these states, phase transitions with spontaneous symmetry breaking (SSB) are expected.

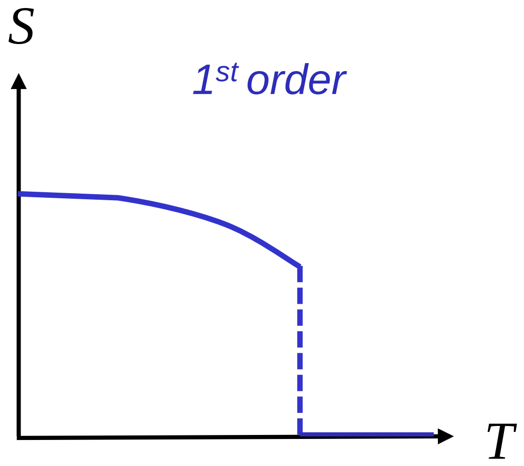
Landau classification

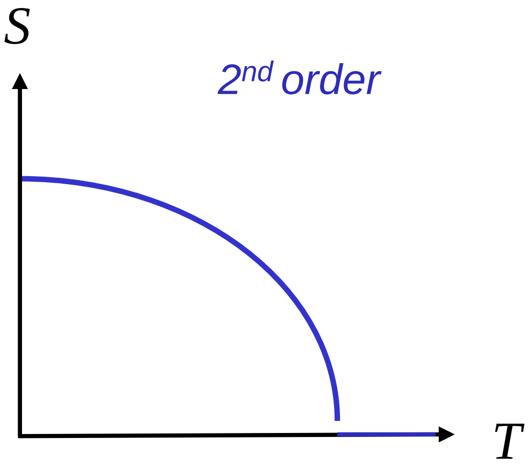
- Depending on the behavior of the order parameter, Landau discriminates between

  1) **First-order (or discontinuous) phase transitions**. The order parameter is not continuous with respect to the control parameter *T*. The correlation length is finite and the two phases coexist at the transition temperature. *Example: gas-liquid, …*
  2) **Second-order (or continuous) phase transitions**. The order parameter is continuous with respect to the control parameter *T*. The correlation length diverges, which forces the whole system to be in a unique phase at the transition. *Example: para-ferro, He I-He II…*

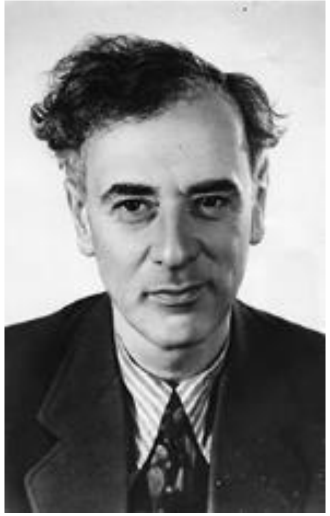


LD Landau





Landau theory: the recipe



LD Landau

1. **Define an order parameter.** For a given system an order parameter must be constructed. This is defined from an averaging procedure = mean field theory, not sensitive to the microscopic details of the system. The order parameter is zero in the disordered phase and non-zero in the ordered phase.

2. **Write a free energy functional.** It is assumed to be an analytic (typically polynomial expansion) function of the order parameter and is constrained by the symmetries of the system. This is the most important part of the theory, wherein most of the physics lies.

3. **Temperature Dependence.** It resides in the lowest order term in the polynomial expansion of the free energy (usually linear). Other terms can be assumed as constants near the phase transition (this is rigorous for a second-order phase transition, and an approximation for a first-order phase transition).

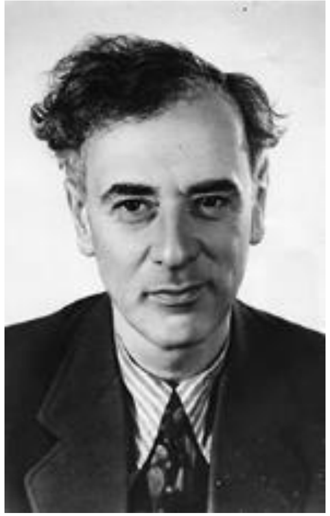
Landau theory (1937)

- At the transition $T=T_c$, if $S\sim 0$, then the free energy density can be power-expanded with respect to the order parameter:

$$f_L(S,T)=f_0(T)+a_1(T)S+\frac{a_2(T)}{2}S^2+\frac{a_3(T)}{3}S^3+\frac{a_4(T)}{4}S^4+\dots$$

coupling to an external field (otherwise vanishes)

quadratic term linear in temperature



LD Landau

- At each temperature, the stable state of the system (and hence the value of S) is determined by the minimization of the Landau free energy.
- Phase transitions occur when the quadratic term in the Landau expansion changes sign whilst all the other coefficients remain positive. If the driving parameter is temperature, the sign-changing term is linear in temperature

$$a_2(T)=a(T-T_c) \qquad a>0$$

2^{nd} order phase transitions

- When odd-terms are removed and for $a_4(T)=b>0,\ \Delta f_L(S,T)=\frac{a(T-T_c)}{2}S^2+\frac{b}{4}S^4+\ldots$

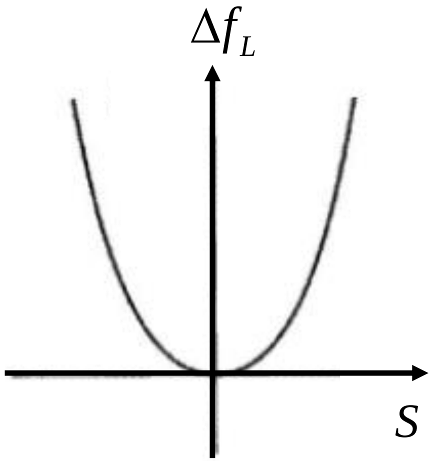
  Minimization of free energy gives $\frac{\partial\Delta f_L}{\partial S}=a(T-T_c)S+bS^3=0$

2nd order phase transitions

- When odd-terms are removed and for $a_4\left(T\right)=b>0,\ \Delta f_L\left(S,T\right)=\frac{a\left(T-T_c\right)}{2}S^2+\frac{b}{4}S^4+...$

Minimization of free energy gives $\frac{\partial \Delta f_L}{\partial S}=a\left(T-T_c\right)S+bS^3=0$

- For $T>T_c$, only one solution: $S=0$ (disordered high temperature phase).



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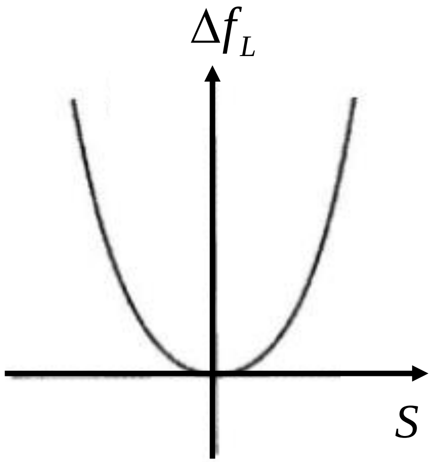

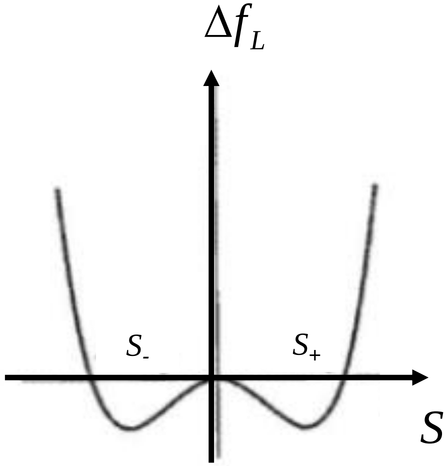

2nd order phase transitions

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  Minimization of free energy gives $\dfrac{\partial \Delta f_L}{\partial S} = a(T-T_c)S + bS^3 = 0$

  - For $T>T_c$, only one solution: $S=0$ (disordered high temperature phase).
  - For $T<T_c$, three roots: $S=0$ (maximum) and $S_\pm = \pm\sqrt{a(T_c - T)/b}$ (minima)

- The order parameter is continuous at $T=T_c$ ⇒ Second-order phase transition:

  - Thermodynamic properties obey **scaling laws** (critical exponents, universality):

$$S \sim |T_c - T|^{\beta} \qquad \xi \sim |T_c - T|^{-\nu} \qquad \tau \sim |T_c - T|^{-\mu} \qquad \mathrm{C} \sim |T_c - T|^{-\alpha} \qquad \ldots$$

  - Large scale fluctuations (the correlation length diverges at the transition)
    ⇒ **Pre-transitional effects**: critical opalescence…
  - First derivatives of free energy are continuous, but higher order ones are not…

1st order phase transitions

- Things are different here as the order parameter vanishes in the disordered phase but is not continuous at T_c (which is marking now the limit of existence of the high temperature phase): hence, *S* is not necessarily small at the transition (unless the transition is weakly first order).

- One still consider the Landau expansion, but with a cubic term

$$\Delta f_L\left(S,T\right)=\frac{a\left(T-T_c\right)}{2}S^2-\frac{c}{3}S^3+\frac{b}{4}S^4+\ldots \qquad a_3\left(T\right)=-c<0$$

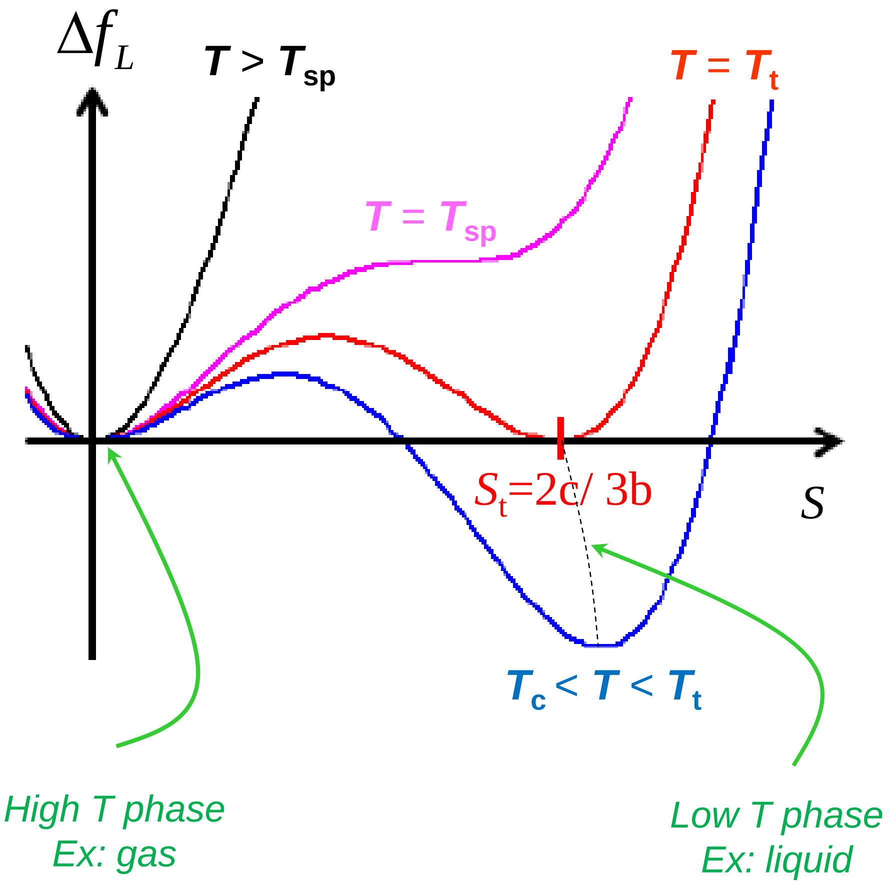
- Two conditions to implement: - minimum of free energy: $\frac{\partial \Delta f_L}{\partial S}=0 \quad \Rightarrow S\left(a\left(T-T_c\right)-cS+bS^2\right)=0$

3 roots for S: 2 min, 1 max when $T\leq T_{sp}=T_c+\frac{c^2}{4ab}$ *spinodal temperature*

- possibility of 2 coexisting phases $\Delta f_L\left(S,T\right)=\Delta f_L\left(0,T\right)=0$

Possible when $T\leq T_t=T_c+\frac{2c^2}{9ab}<T_{sp}$ *temperature of the transition*

1^{st} order phase transitions



- There is a jump of the order parameter at the transition

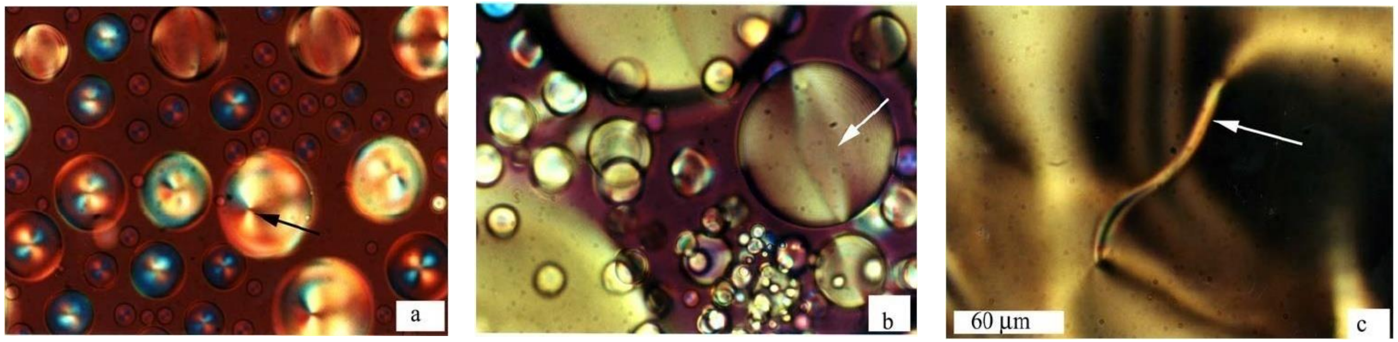
⇒ First-order phase transition:

  - First derivatives of free energy are not continuous ⇒ **Latent heat.**
  - Scaling laws.
  - Coexistence of the two phases (nucleation).

Remark: A first-order transition does not require a change of symmetry (ex: water-steam).

What about the IN transition ?

M Kleman et al. Phil Mag 86, 4117 (2006)



1- Spherulites formation = nucleation

2- Growth of domains

3- Threads when domains mingle ??

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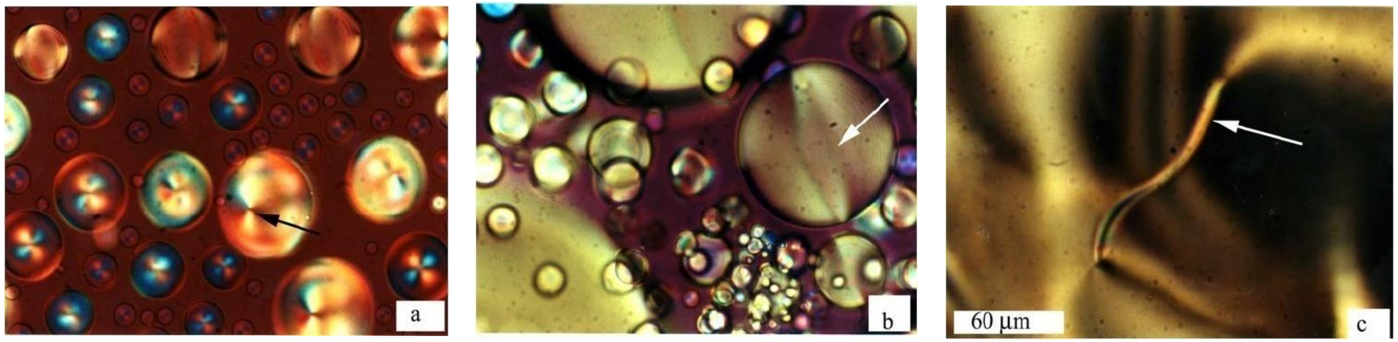

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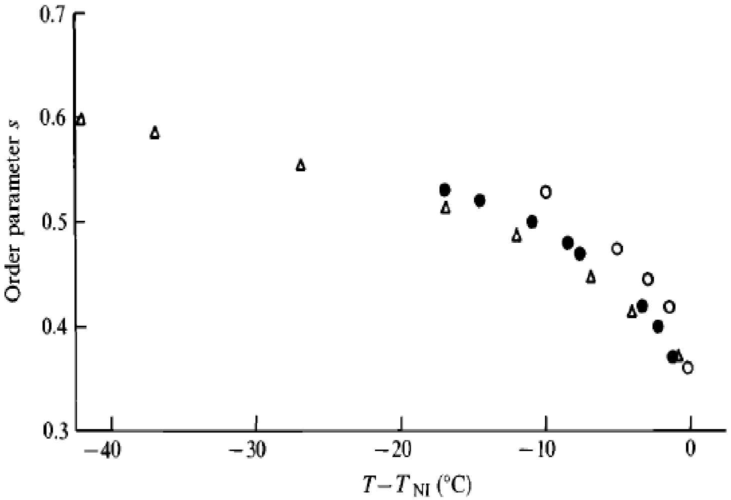

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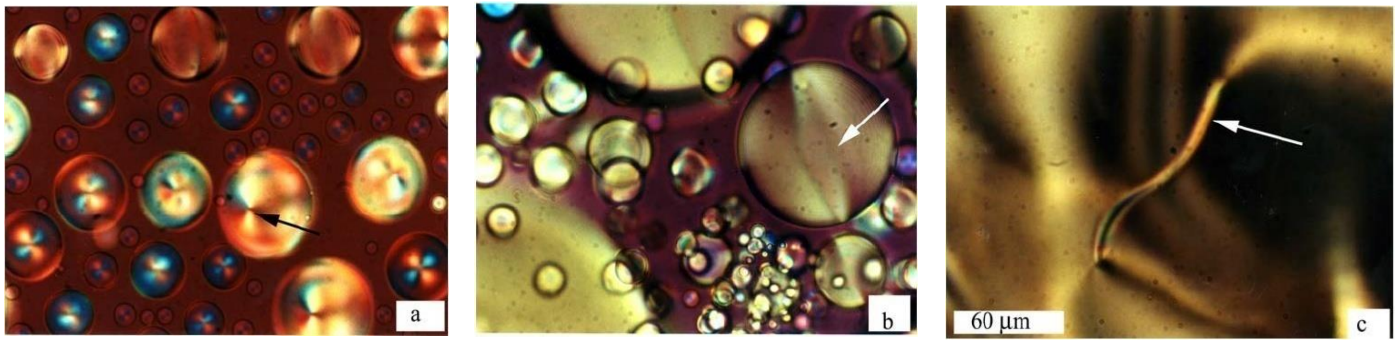

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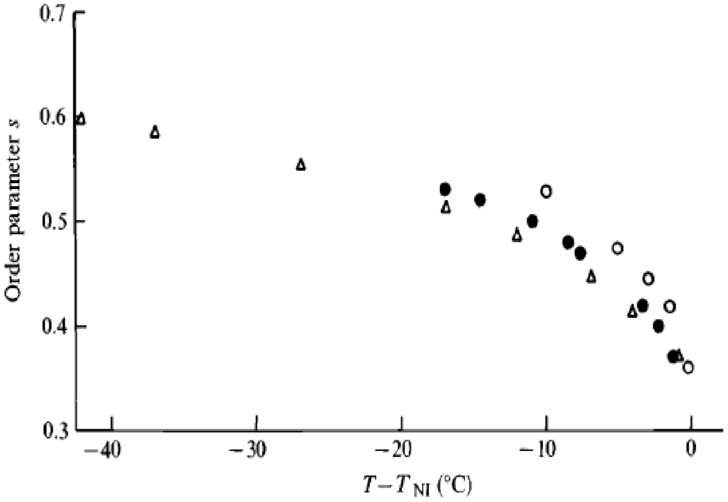

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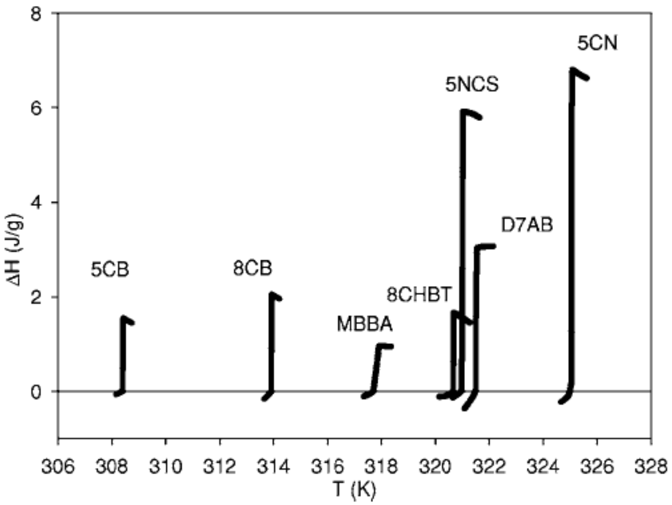

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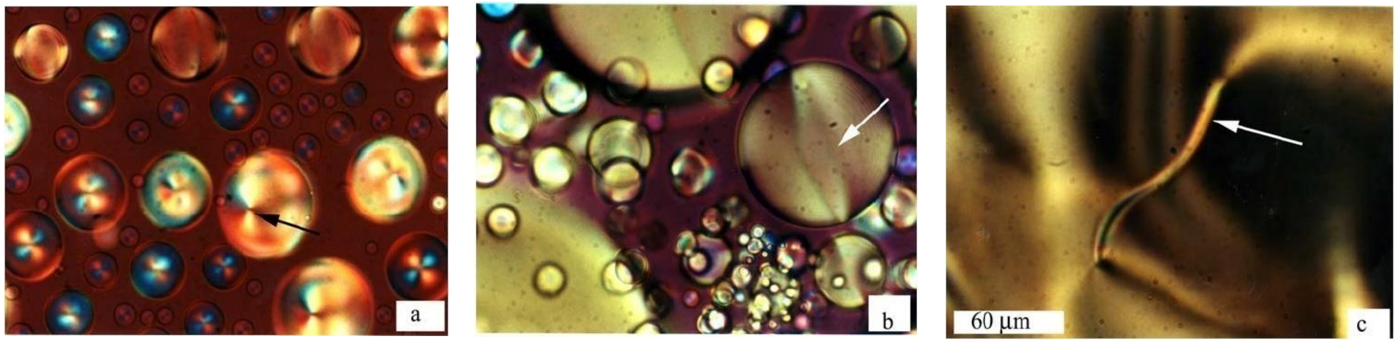

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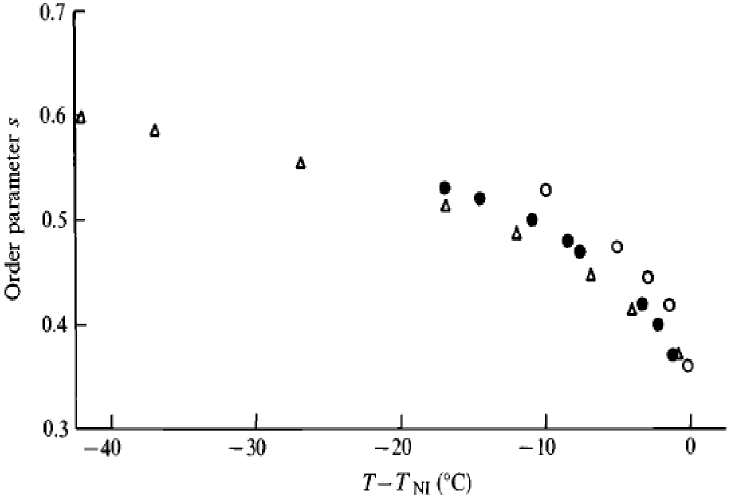

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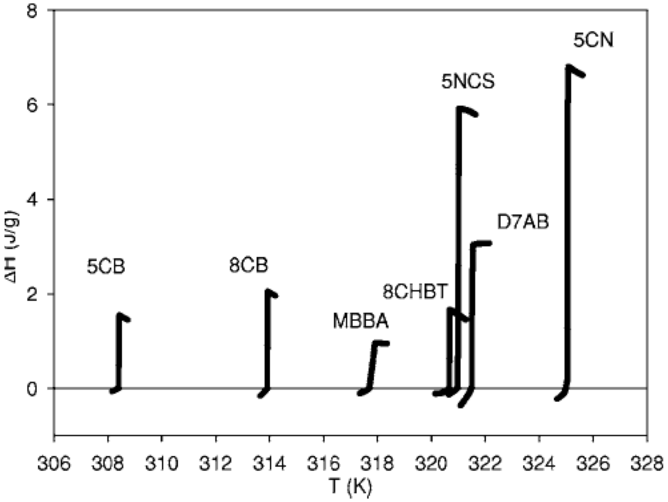
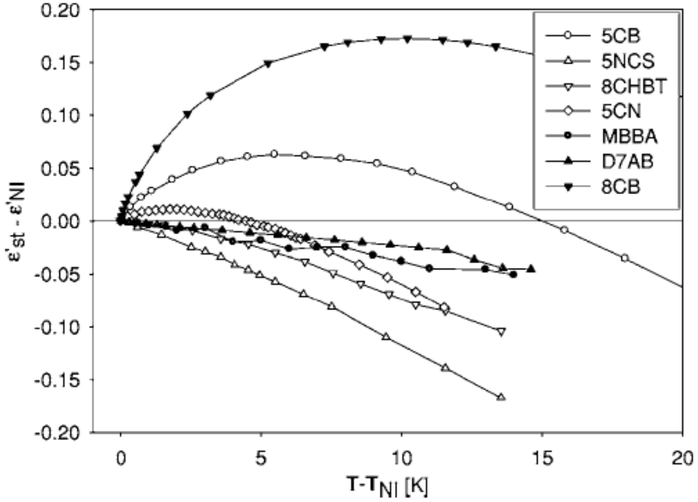


… and pretransitional effects



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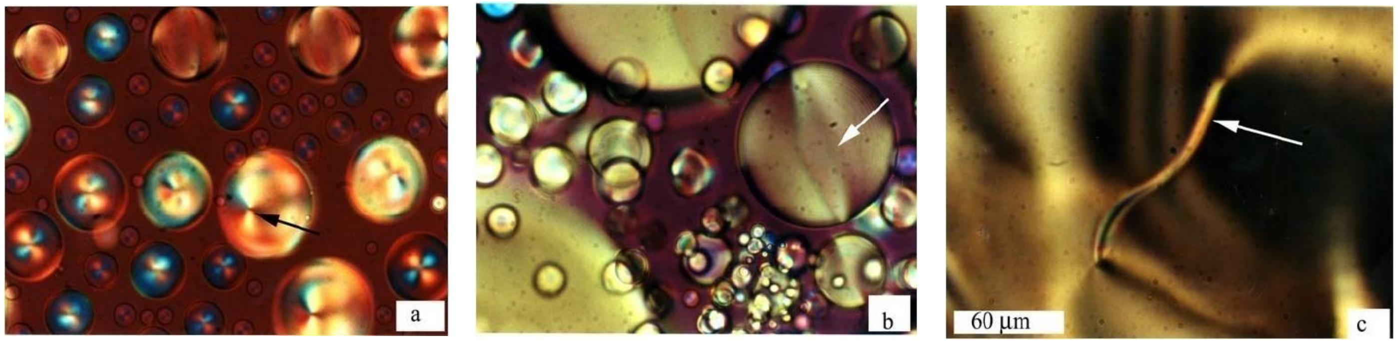

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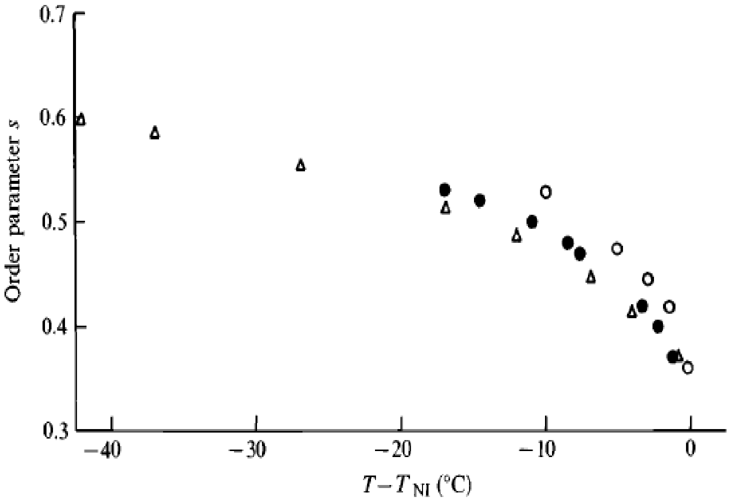

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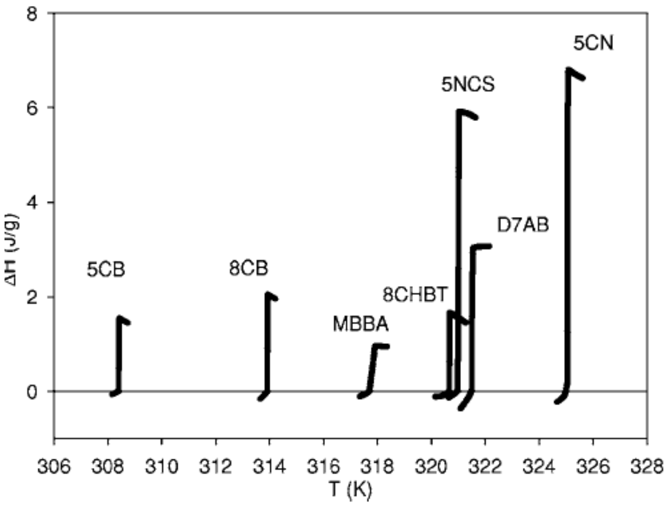

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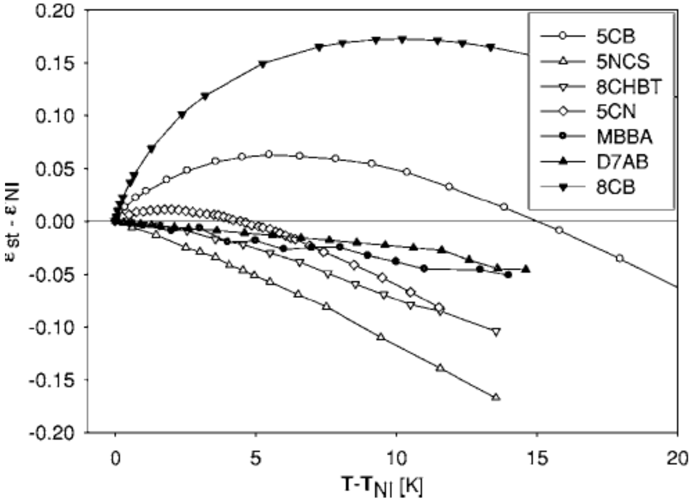

The isotropic-nematic phase transition is nearly 2^{nd} order.

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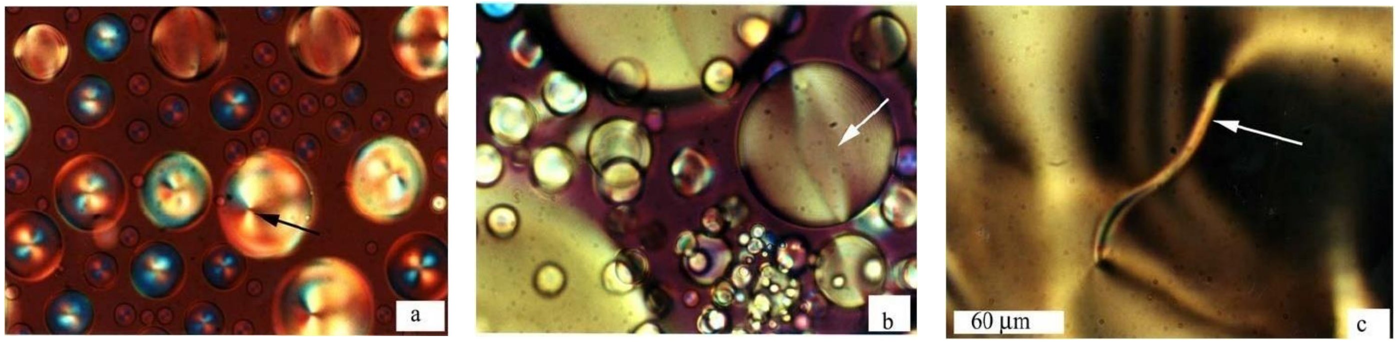

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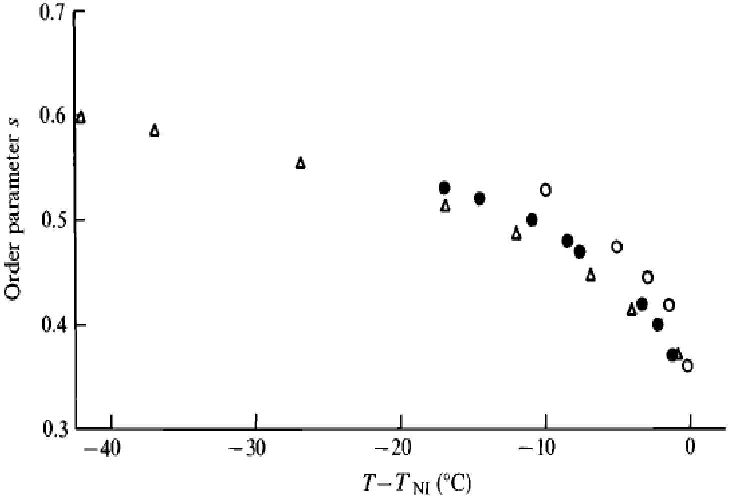

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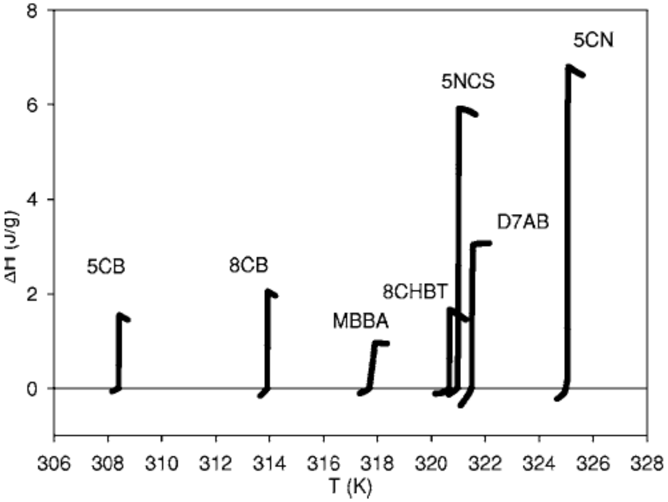

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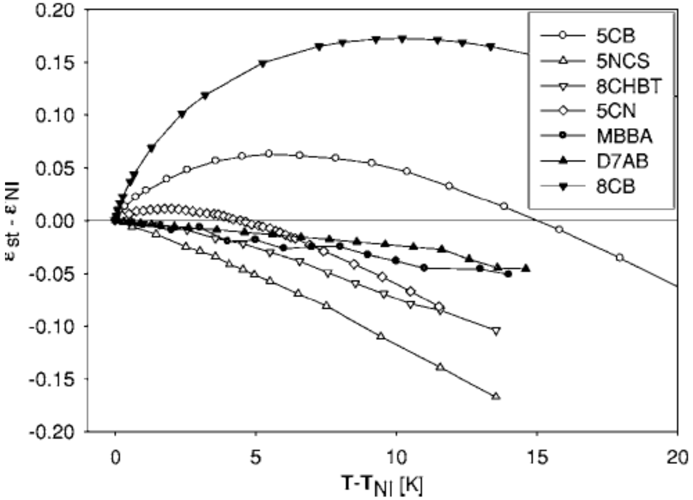

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A few definitions

- **Topology** is the branch of mathematics focusing on the properties that remain unchanged when a topological space is “smoothly deformed” (neither torn apart nor punctured). When a topological property changes, it occurs by integer steps, not gradually. In practice, we will consider manifolds for which each point of the topological space locally looks like $\mathbb{R}^n$.

The Nobel Prize in Physics 2016

Topology – what is it?

Topologogy is a field of mathematics that describes properties that are stable and only change in integer steps: 1, 2, 3...

The number of holes is a topological invariant that is always an integer, but never anything in between.

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- **Algebraic topology** (Poincaré’s former *analysis situs*) seeks algebraic invariants (numbers, abelians groups, rings…) to study and classify topological spaces into equivalence classes. These objects may characterize the connectedness, the number of holes, the existence of boundaries...

1 hole

$\not\Leftrightarrow$

3 holes

A few definitions

- Two manifolds are topologically equivalent or **homeomorphic** if there exists a bijective and continuous map between them. These two manifolds have to be of the same dimension (Brouwer's invariance of domain theorem). Intuitively, it corresponds to a continuous deformation with no gluing or tearing.

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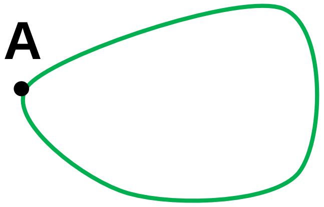
- **Homotopy** provides a weaker notion of equivalence between topological spaces than homeomorphism. It corresponds to a continuous deformation where bijectivity is not preserved, i.e. gluing, shrinking or fattening the space is allowed: for instance, a loop (dimension 1) is homotopic to a point (dimension 0).



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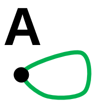

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The fundamental group

- The first homotopy group (or Poincaré group or fundamental group) $\pi_1(M)$ tests if all closed curves on a manifold M are homotopic to a point. If it is the case, the group is trivial and $\pi_1(M)=I$. When does this fail ?

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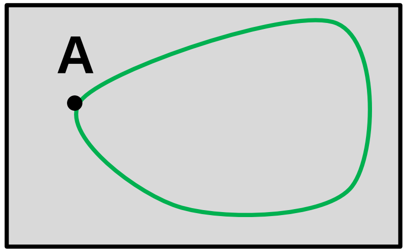
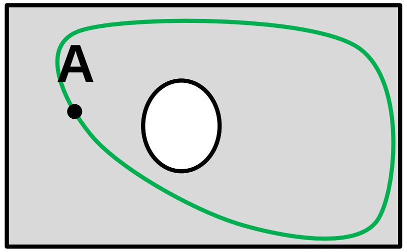
$M=\mathbb{R}^2$





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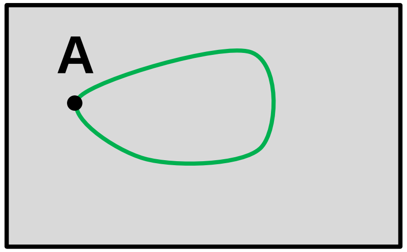

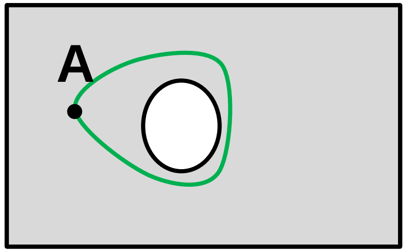

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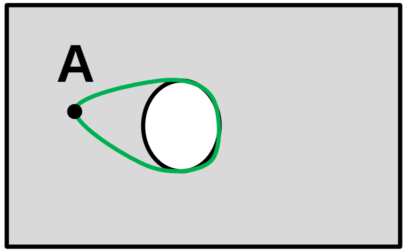

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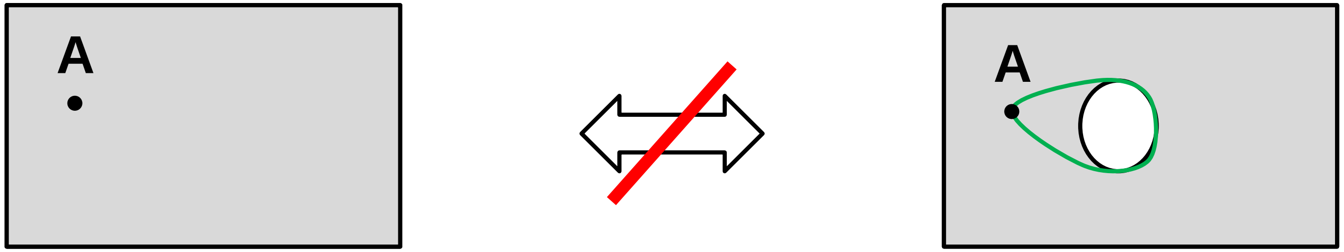


simply-connected

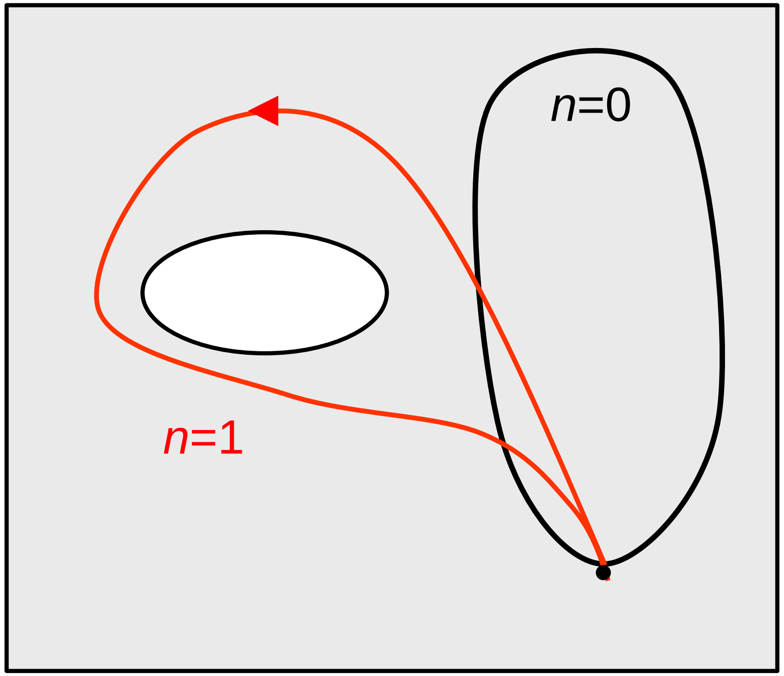
not simply-connected

The first homotopy group also tests the simply-connectedness of the manifold: in 2D, this is equivalent to test the existence of a 0D-hole.

The fundamental group

- When $\pi_1(M)\neq$I, there are different families (« equivalence classes ») of homotopic loops sharing the same **winding number** (Whitney-Graustein theorem). The winding number of a regular curve is the number of times the tangent vector fully rotates counterclockwise when going once around the curve.

$M=\mathbb{R}^2$



Non-homotopic loops

The fundamental group

- When $\pi_1(M)\neq I$, there are different families (« equivalence classes ») of homotopic loops sharing the same **winding number** (Whitney-Graustein theorem). The winding number of a regular curve is the number of times the tangent vector fully rotates counterclockwise when going once around the curve.

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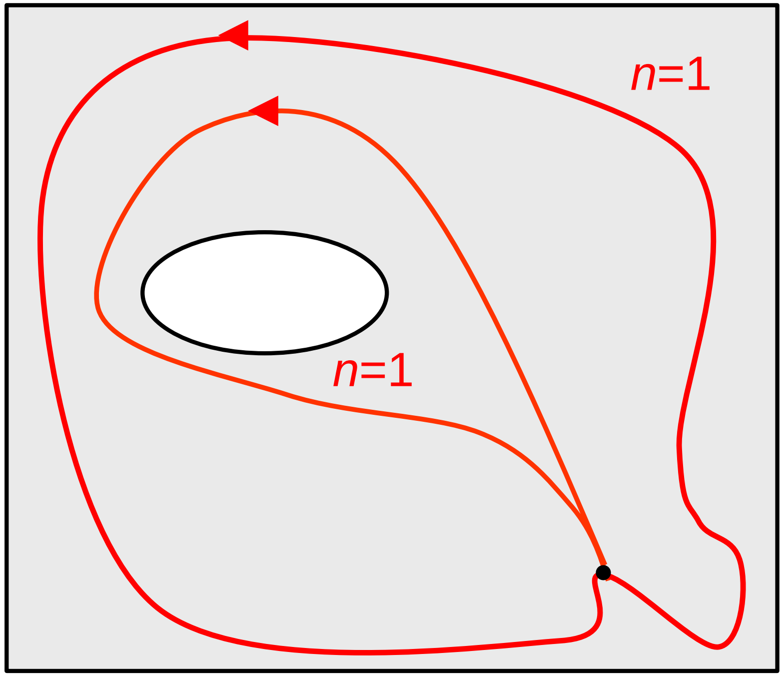

Homotopic loops of the same equivalence class

The fundamental group

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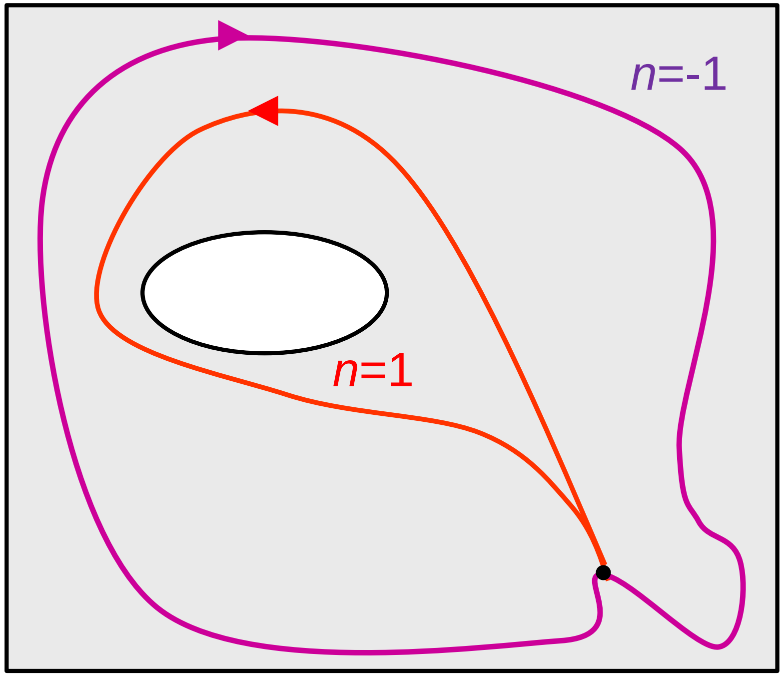
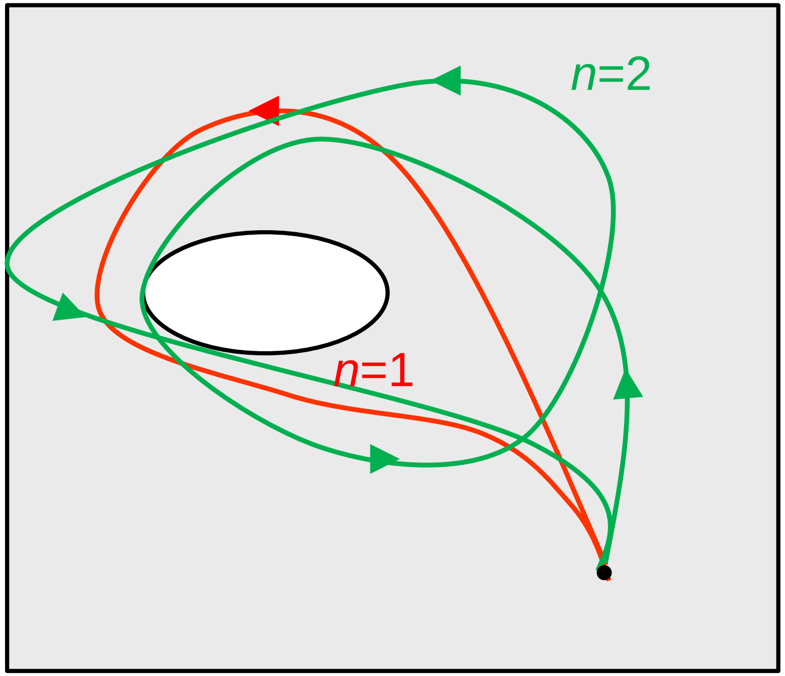


Homotopic loops of different equivalence classes

The fundamental group

- When $\pi_1(M)\neq$I, there are different families (« equivalence classes ») of homotopic loops sharing the same **winding number** (Whitney-Graustein theorem). The winding number of a regular curve is the number of times the tangent vector fully rotates counterclockwise when going once around the curve.

$M=\mathbb{R}^2$

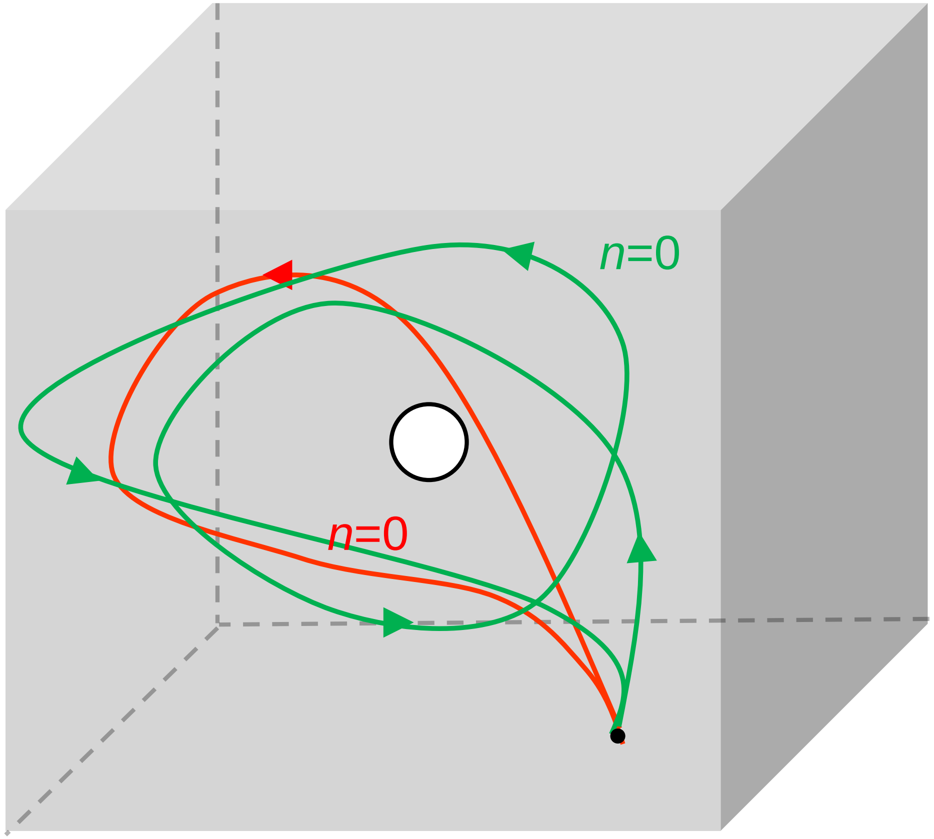


Homotopic loops of different equivalence classes

The fundamental group

- In 3D, testing $\pi_1(M)$ means: can you lasso something ?

$M=\mathbb{R}^3$



▶ The answer is no: $\pi_1(M)=\mathrm{I}$

The fundamental group

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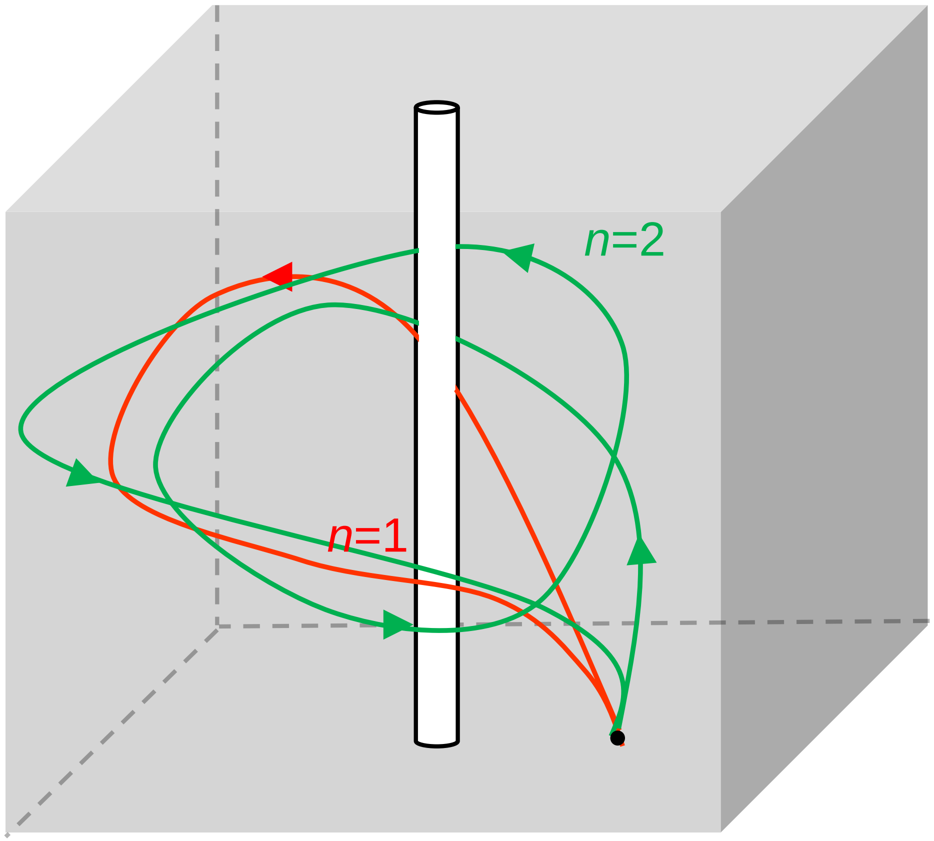

- ▶ The answer is yes: $\pi_1(M) \neq \mathrm{I}$

The fundamental group

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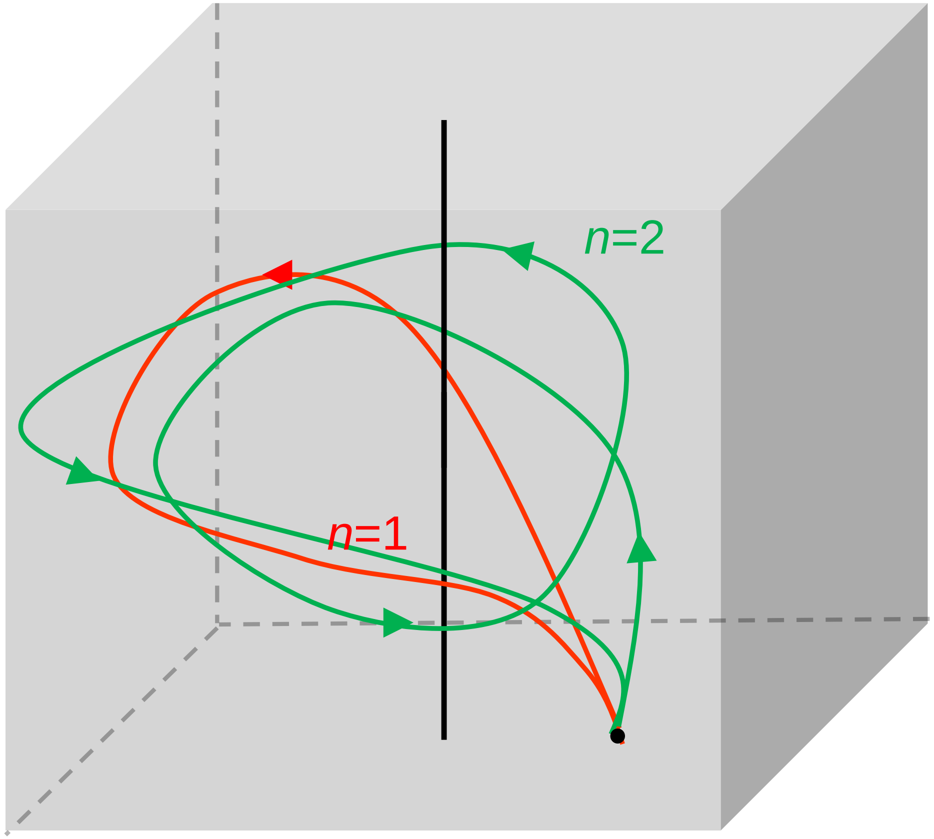


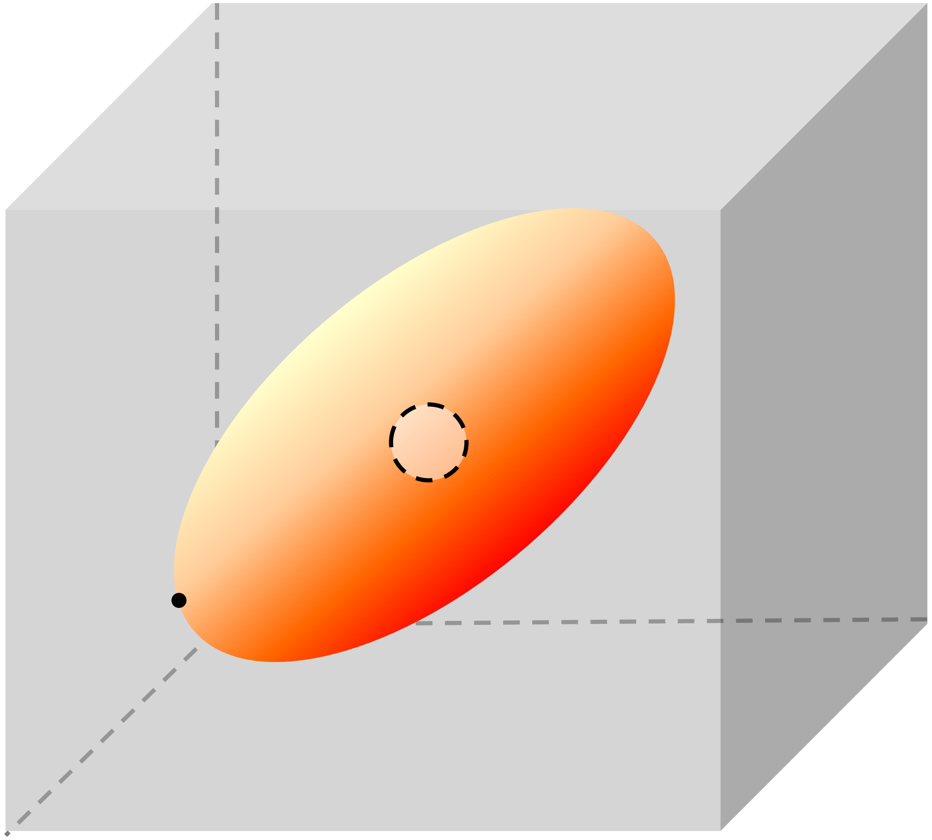
▶ The answer is yes: $\pi_1(M)\neq \mathrm{I}$

In 3D, the first homotopy group
tests the existence of 1D-hole.
(cf Aharonov-Bohm phase)

Other homotopy groups

- Similarly, $\pi_2(M)$ tests if all closed surfaces on a manifold M are homotopic to a point.

$M=\mathbb{R}^3$



Other homotopy groups

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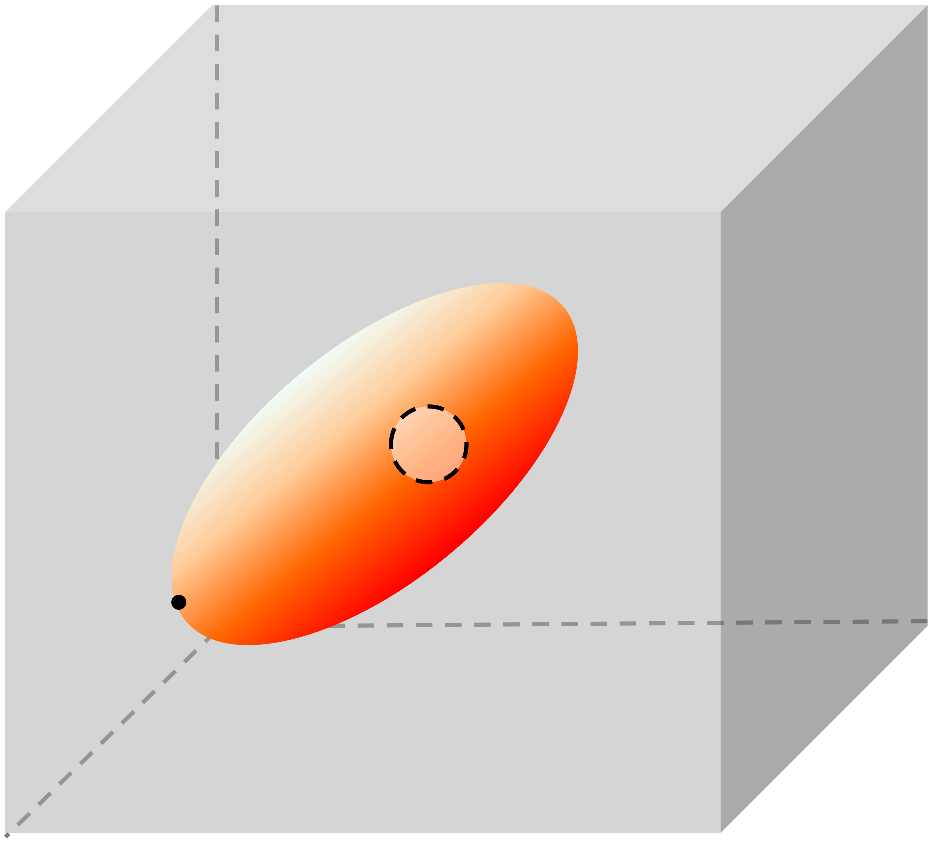

Other homotopy groups

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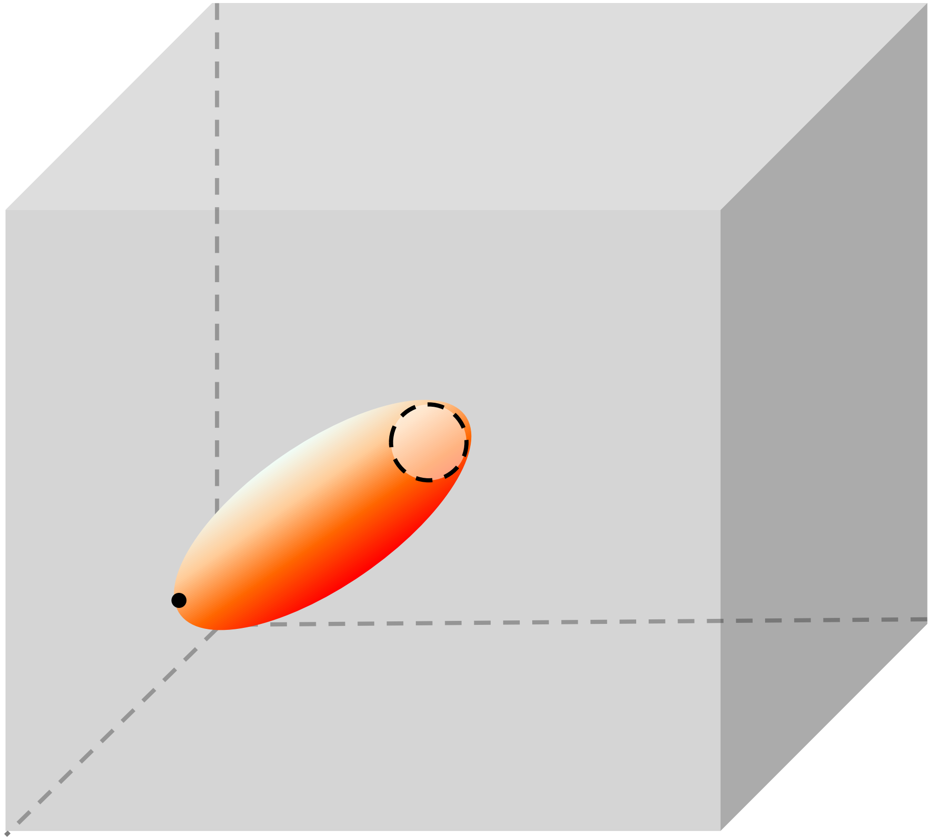
$M=\mathbb{R}^3$

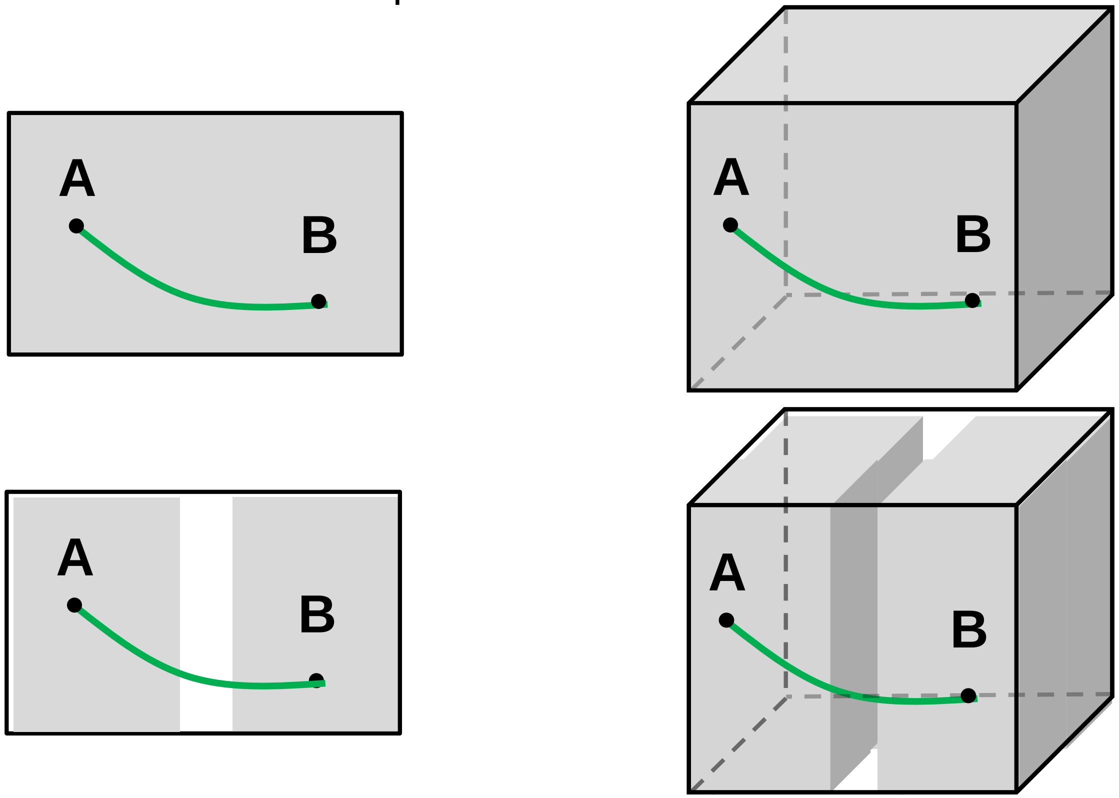


$\pi_2(M)\neq$ I

In 3D, the second homotopy group tests the existence of 0D-hole.

Other homotopy groups

- $\pi_0(M)$) tests if the topological space is (pathwise) **connected**, i.e. if for any pair of points, one can find a path between them that remains in the topological space. Intuitively, it corresponds to the notion of a space that is in one whole piece.



Connected

Not connected

1D-hole

2D-hole

Use for phase transitions

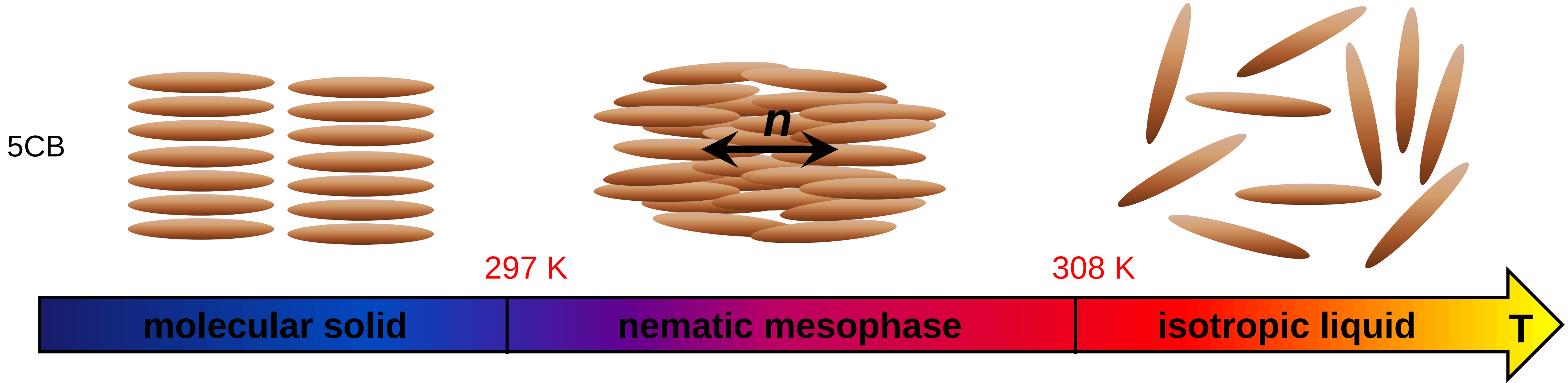
The bare minimum

- In practice, homotopy is mostly of matter of counting holes… but p-holes in n-dimensional manifolds and it answers the question: what kinds of singularities=defects are there ?
- In dimension n, if the kth homotopy group $\pi_k(M) \neq I$, then topologically defects of dimension n-1-k appear.

Connection with spontaneous symmetry breaking

During a phase transition with a symmetry breaking pattern G → H, defects arise according to the topology of the ***order parameter space*** $M=G/H$.

Application to the IN transition



orientational order	↔	***orientational order***	SSB	*no orientational order*
positional order	SSB	*no positional order*	↔	*no positional order*

symmetry group	***space group G_E***	***O(2) x T(3)***	***SO(3) x T(3)***

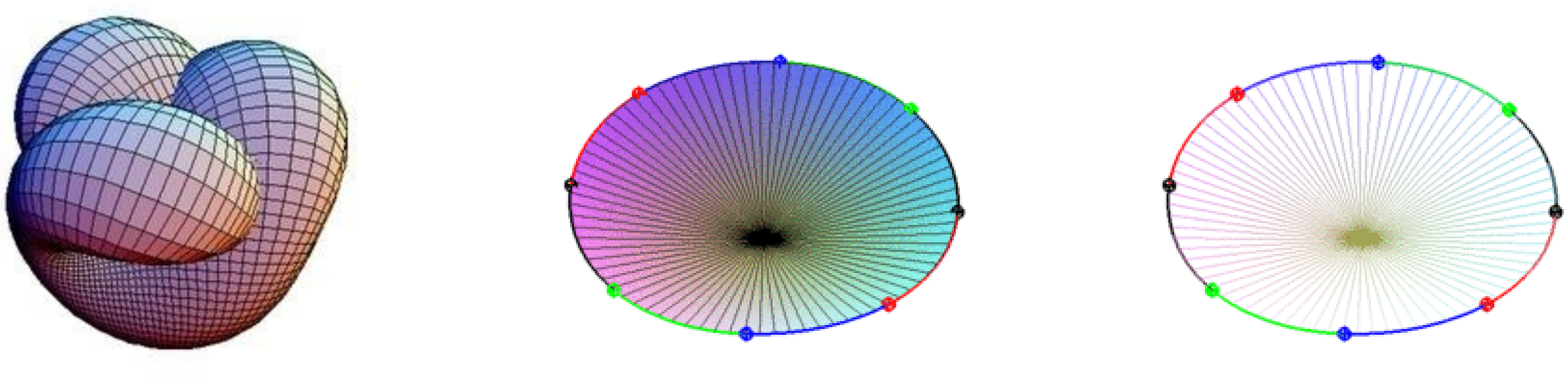
⇒ For the isotropic-nematic phase transition, ***G=SO(3) and H=O(2)***

Topology of Boy surface

- For the isotropic-nematic phase transition, the symmetry-breaking pattern is $SO(3) \rightarrow O(2)$

$\Rightarrow$ $M = SO(3)/O(2) \sim S^2/Z_2$ manifold=« Boy surface »

- Immersion of the real projective plane in 3-dimensional space



⇨ *How to determine the topology of something like like this ?*

Topology of Boy surface

▶ The Euler-Poincaré characteristic in 3 definitions

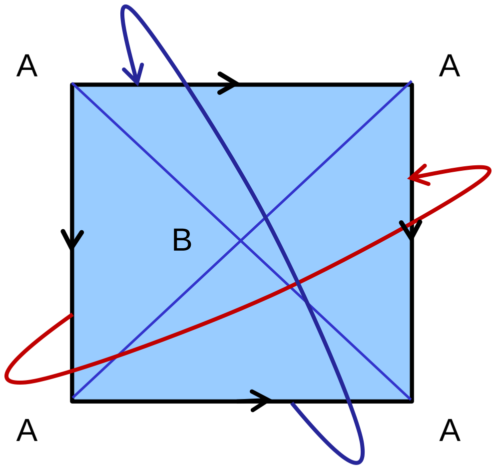
- *Triangulation and polygons* $\boxed{\chi\text{=F-A+S}}$

- *Gauss-Bonnet theorem* $$\boxed{\chi = \frac{1}{2\pi}\oiint_{torus} FdS}$$

- *Genus = number of holes: (orientable surface)* $\boxed{\chi\text{=2-2g}}$

Topology of Boy surface

▶ The Euler-Poincaré characteristic in 3 definitions



- *Triangulation and polygons* $\boxed{\chi\text{=F-A+S}}$ χ=4-6+2=0

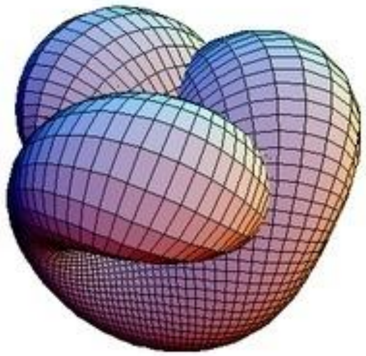
- *Gauss-Bonnet theorem* $$\boxed{\chi = \frac{1}{2\pi}\oiint_{torus} FdS}$$

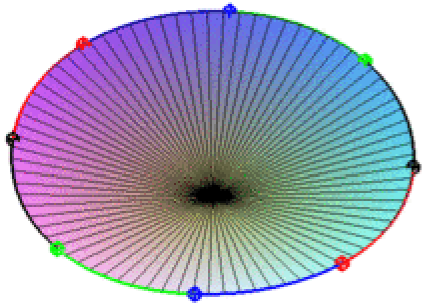
$$\boldsymbol{OM} = \left(\left[R + r\cos\theta\right]\cos\varphi, \left[R + r\cos\theta\right]\sin\varphi, r\sin\theta\right) \Rightarrow F\left(r,\theta\right) = \frac{\cos\theta}{r\left(R + r\cos\theta\right)} \Rightarrow \chi = 0$$

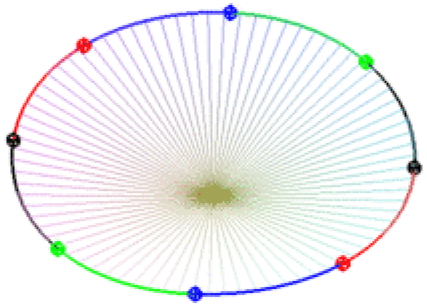
- *Genus = number of holes:*
  (orientable surface) $\boxed{\chi\text{=2-2g}}$ χ=0 $\Rightarrow$ g=1 $\pi_1(T^2)=\mathbb{Z}^2$

Topology of Boy surface

- Immersion of the real projective plane in 3-dimensional space







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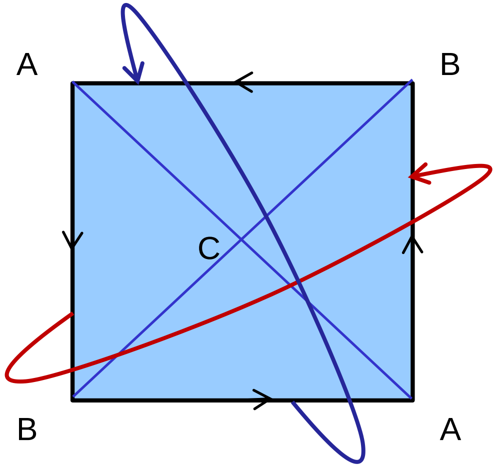


Poincaré-Euler characteristic χ=F-A+S=4-6+3=1

Genus = number of holes:
(non-orientable surface) $g = 2 - \chi$=1

1 hole ⇒ non-trivial fundamental group

Topology of Boy surface

L Michel. Rev Mod Phys 52(3) (1980)

- **Content of homotopy groups**

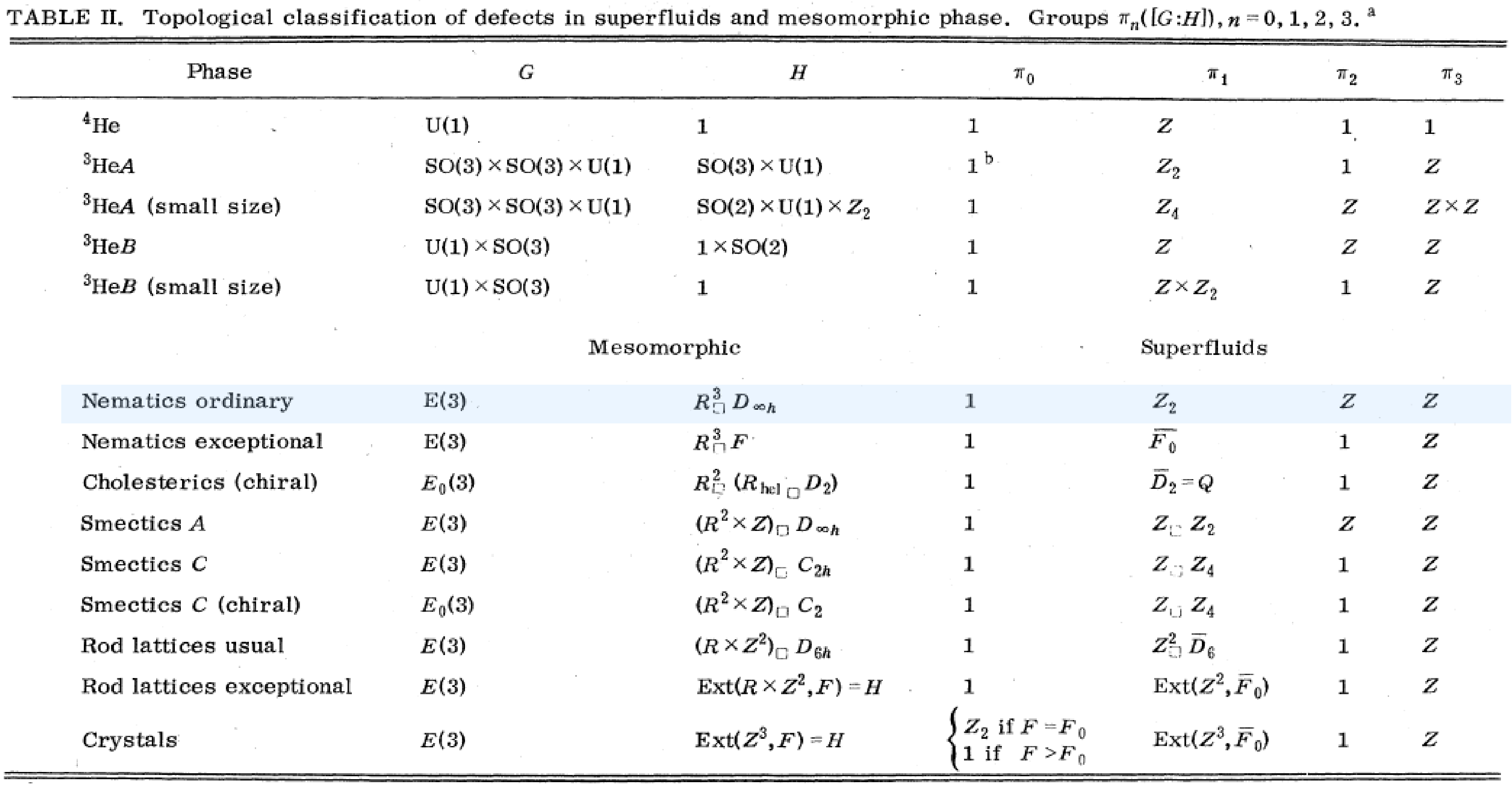


TABLE II. Topological classification of defects in superfluids and mesomorphic phase. Groups $\pi_n([G:H]), n=0,1,2,3.$ [a]

Phase	G	H	π_0	π_1	π_2	π_3
^{4}He	U(1)	1	1	Z	1	1
^{3}HeA	SO(3) × SO(3) × U(1)	SO(3) × U(1)	1 [b]	Z_2	1	Z
^{3}HeA (small size)	SO(3) × SO(3) × U(1)	SO(2) × U(1) × Z_2	1	Z_4	Z	$Z \times Z$
^{3}HeB	U(1) × SO(3)	1 × SO(2)	1	Z	Z	Z
^{3}HeB (small size)	U(1) × SO(3)	1	1	$Z \times Z_2$	1	Z
		Mesomorphic		Superfluids		
Nematics ordinary	E(3)	$R^3_\square D_{\infty h}$	1	Z_2	Z	Z
Nematics exceptional	E(3)	$R^3_\square F$	1	$\overline{F}_0$	1	Z
Cholesterics (chiral)	$E_0(3)$	$R^2_\square (R_{\text{hel}}\,_\square D_2)$	1	$\overline{D}_2 = Q$	1	Z
Smectics A	$E(3)$	$(R^2 \times Z)_\square D_{\infty h}$	1	$Z_\square Z_2$	Z	Z
Smectics C	$E(3)$	$(R^2 \times Z)_\square C_{2h}$	1	$Z_\square Z_4$	1	Z
Smectics C (chiral)	$E_0(3)$	$(R^2 \times Z)_\square C_2$	1	$Z_\square Z_4$	1	Z
Rod lattices usual	$E(3)$	$(R \times Z^2)_\square D_{6h}$	1	$Z^2_\square \overline{D}_6$	1	Z
Rod lattices exceptional	$E(3)$	$\text{Ext}(R \times Z^2, F) = H$	1	$\text{Ext}(Z^2, \overline{F}_0)$	1	Z
Crystals	$E(3)$	$\text{Ext}(Z^3, F) = H$	$\begin{cases} Z_2 \text{ if } F = F_0 \\ 1 \text{ if } F > F_0 \end{cases}$	$\text{Ext}(Z^3, \overline{F}_0)$	1	Z

Topology of Boy surface

- **Content of homotopy groups**

$$\pi_0(\boldsymbol{M}) = \mathrm{I}$$

No domain wall

Topology of Boy surface

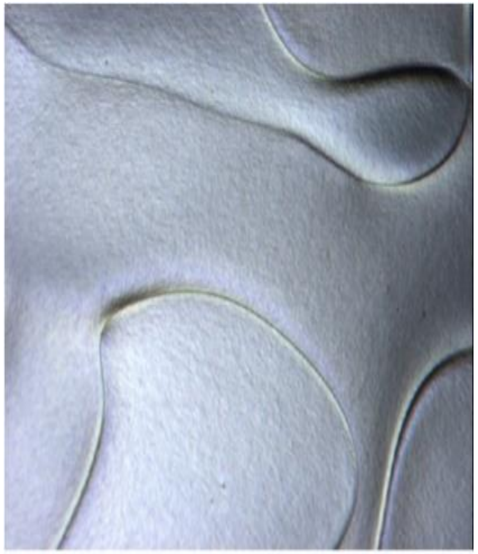
- **Content of homotopy groups**

$\pi_0(\boldsymbol{M})= \mathrm{I}$

$\pi_1(\boldsymbol{M})=\mathbf{Z}_2$

No domain wall

Line defects: disclinations, loops…



P. Pieranski, Paris Orsay.

Topology of Boy surface

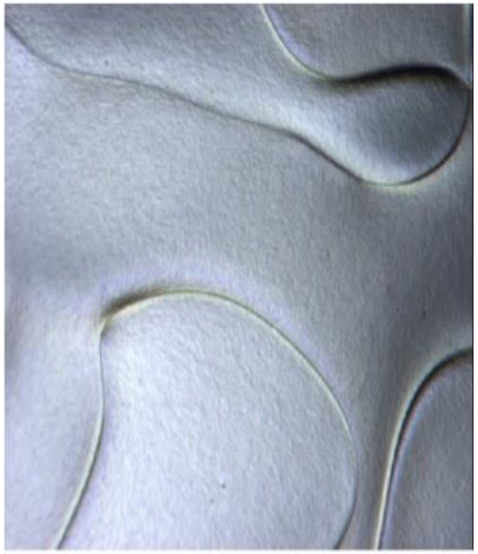
▶ Content of homotopy groups

$\pi_0(M)= I$

No domain wall

$\pi_1(M)=Z_2$

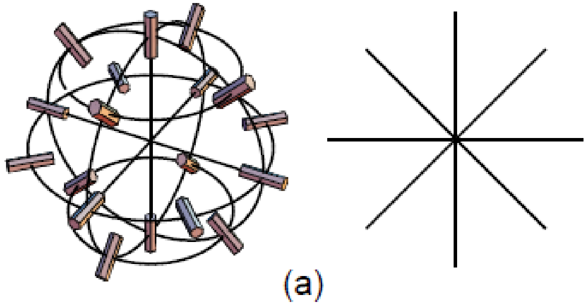
Line defects: disclinations, loops…



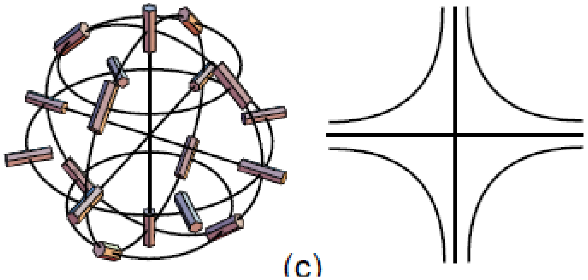
P. Pieranski, Paris Orsay.

$\pi_2(M)=\mathbb{Z}$

Monopoles: radial and hyperbolic hedgehogs…



(a)



(c)

Topology of Boy surface

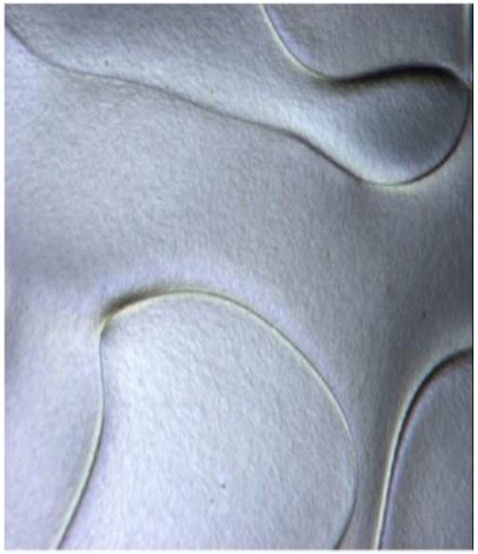
Content of homotopy groups

$\pi_0(M) = I$

No domain wall

$\pi_1(M) = Z_2$

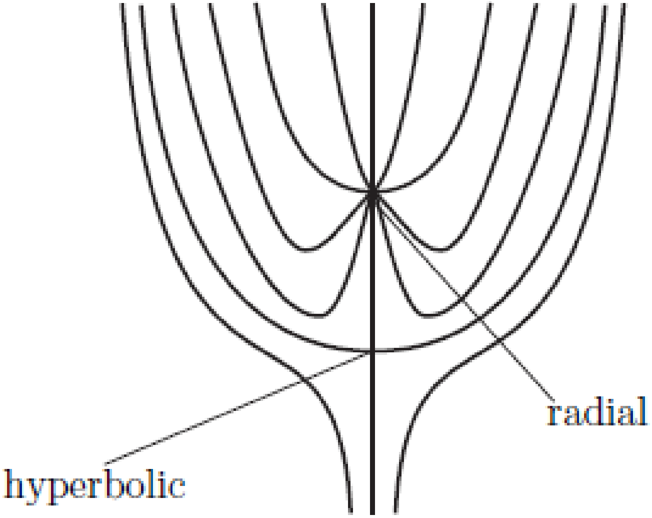
Line defects: disclinations, loops…



P. Pieranski, Paris Orsay.

$\pi_2(M) = \mathbb{Z}$

Monopoles: radial and hyperbolic hedgehogs…



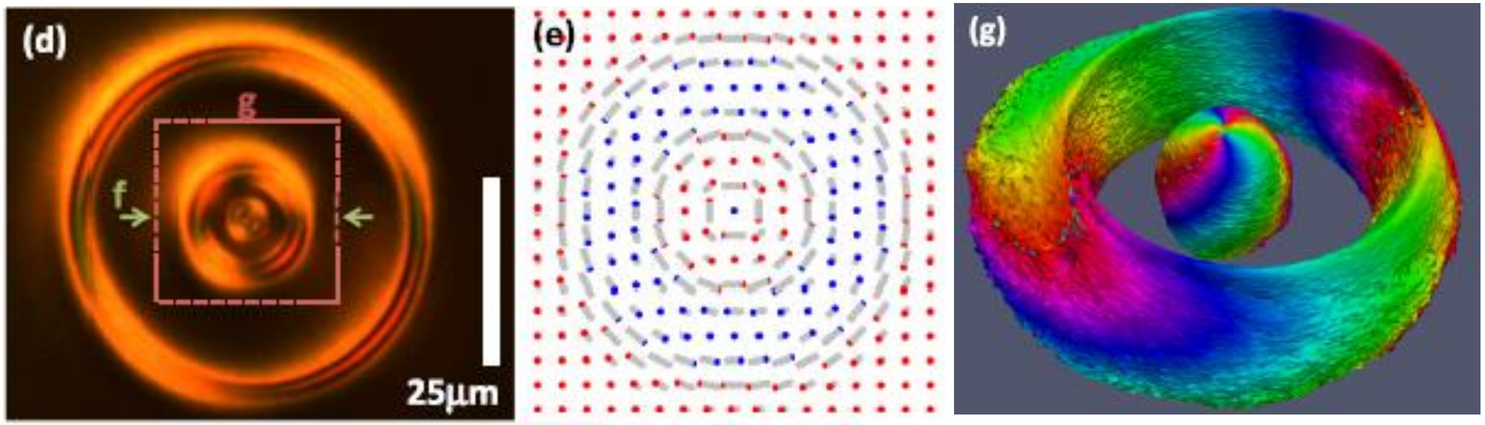
Topology of Boy surface

BG Chen et al. PRL 110, 237801 (2013)
PJ Ackerman et al. PRE 90 (2014)

▶ Content of homotopy groups

$\pi_3(M)=\mathbb{Z}$ *Textures: **Skyrmions**, Hopf fibrations…*

- Hypothetic topologically stable configuration (soliton) of the pion field, predicted in the context of Quantum Chromodynamics (Tony Skyrme, 1962). Also reported in solid-state physics (spintronics, superconductivity…), Bose-Einstein condensates… and of course in soft-matter.
- Topological number $N_s = \frac{1}{4\pi}\iint d^2\mathbf{r}\,\mathbf{n}\left(\frac{\partial\mathbf{n}}{\partial x}\frac{\partial\mathbf{n}}{\partial y}\right)$

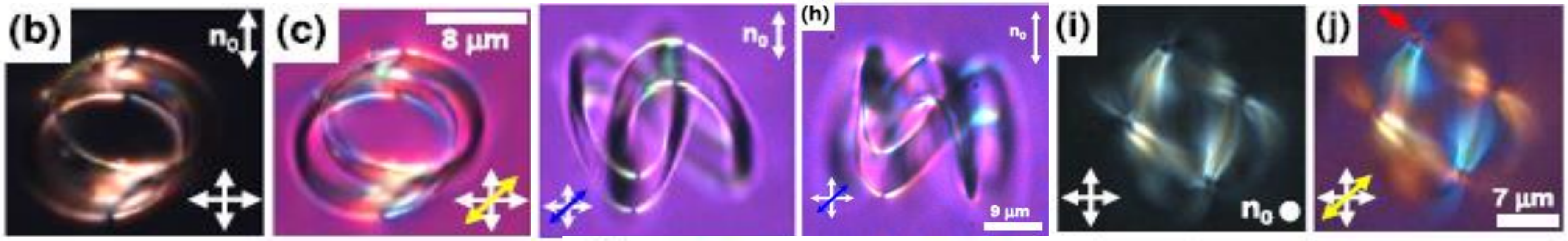


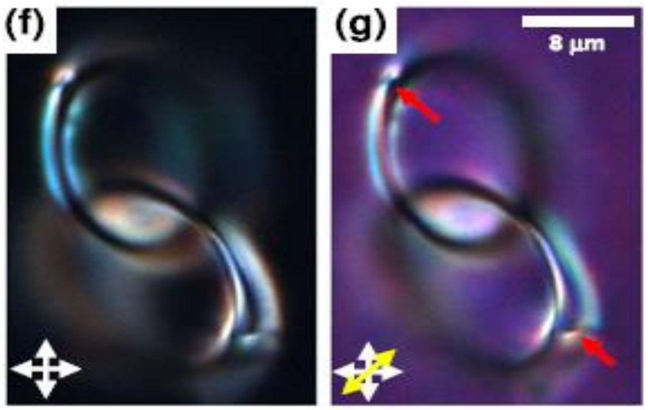
Topology of Boy surface

II Smalyukh Rep Prog Phys 83 (2020)

- **Content of homotopy groups**

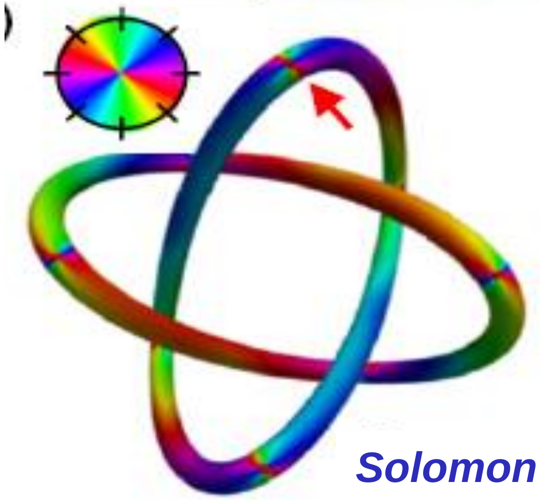
$\pi_3(M)=\mathbb{Z}$ *Textures: Skyrmions, **Hopf fibrations**…*





Hopf link

Fourfold -Knot



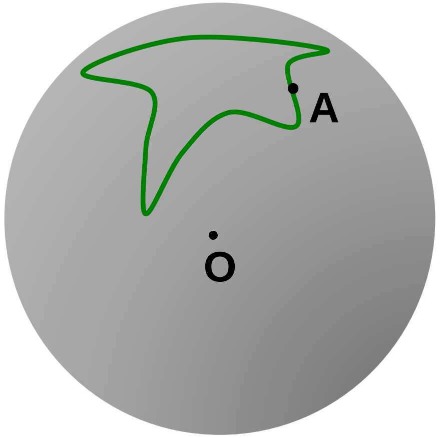
Solomon link

Content of π_1

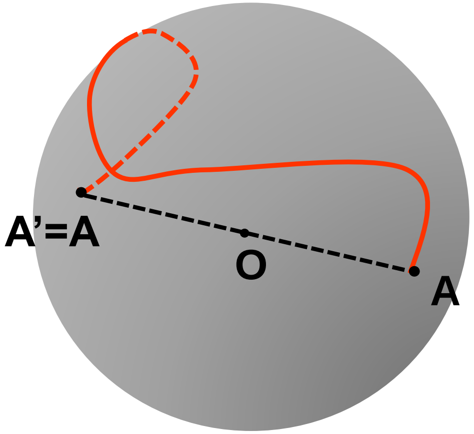
- **What does Z_2 mean for line defects?**

There are only two equivalence classes for the linear defects:

unstable



$$\pi_1(\mathbb{R}P^2)=Z_2=\{0,1\}$$



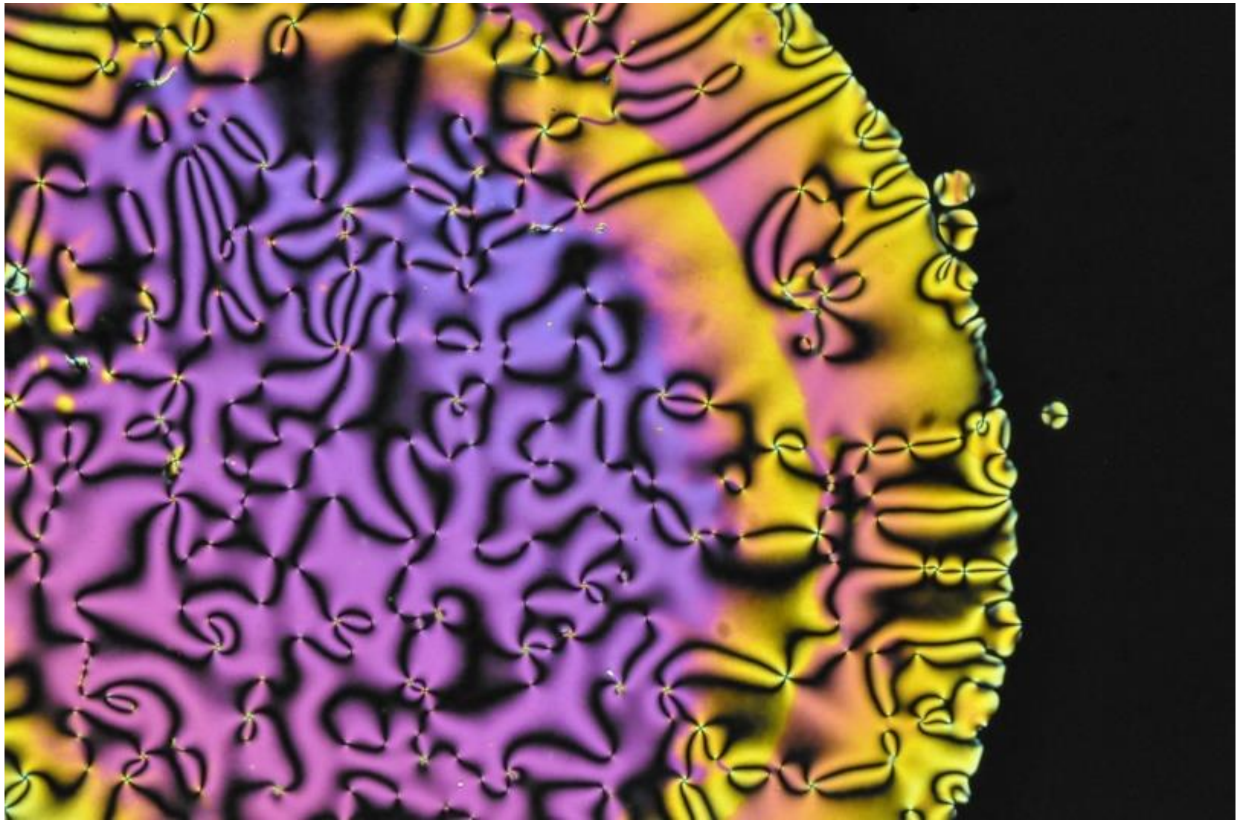
stable

These two classes of defects can be combined according to the algebra of $\mathbb{Z}/2\mathbb{Z}$:

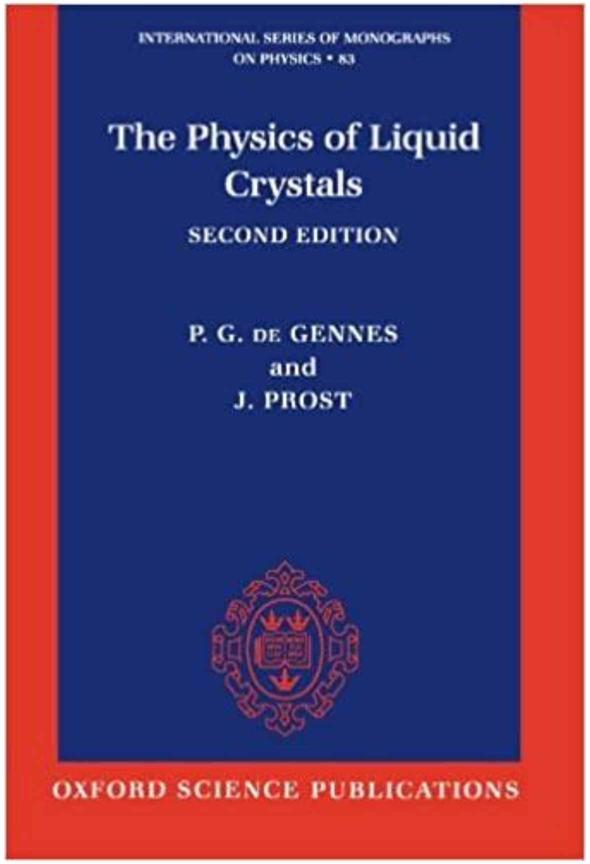
0+0=0 **0+1=1** **1+1=0**

Next lecture…

- **A riddle: what are these ?**



Bibliography



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The Physics of Liquid Crystals

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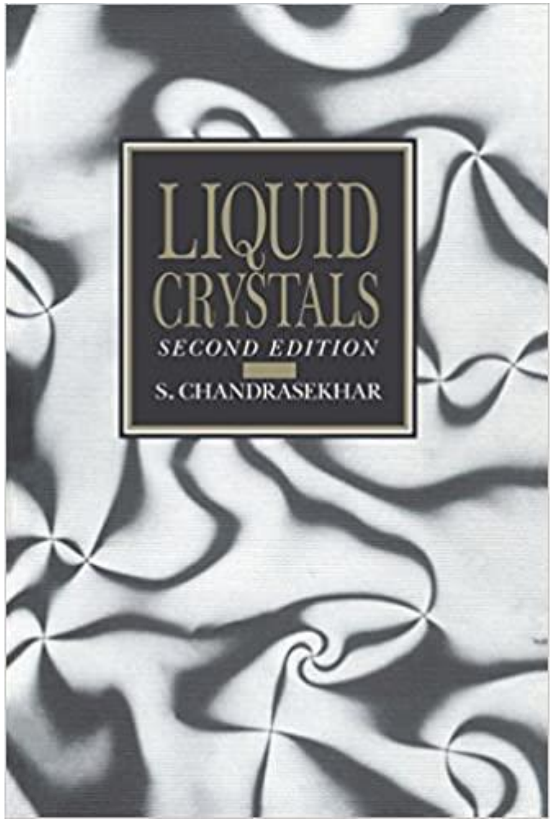
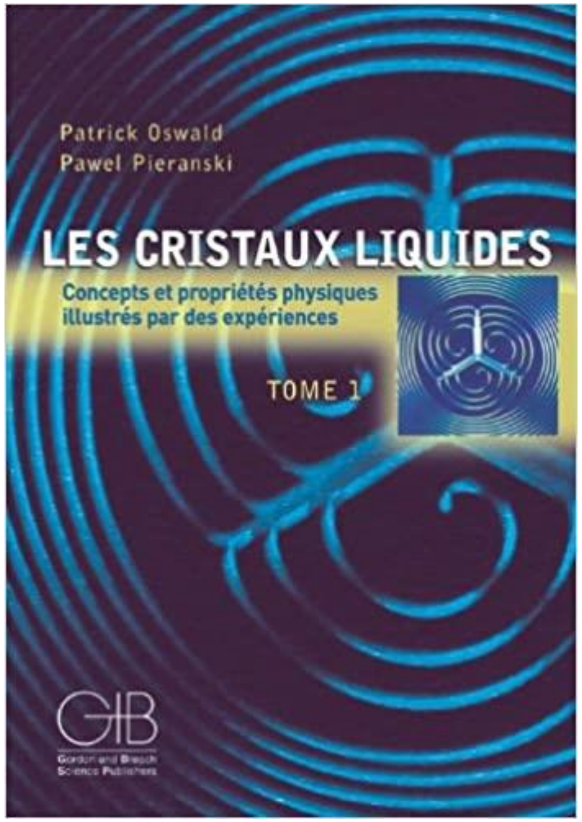


LIQUID CRYSTALS

SECOND EDITION

S. CHANDRASEKHAR



Patrick Oswald
Pawel Pieranski

LES CRISTAUX LIQUIDES

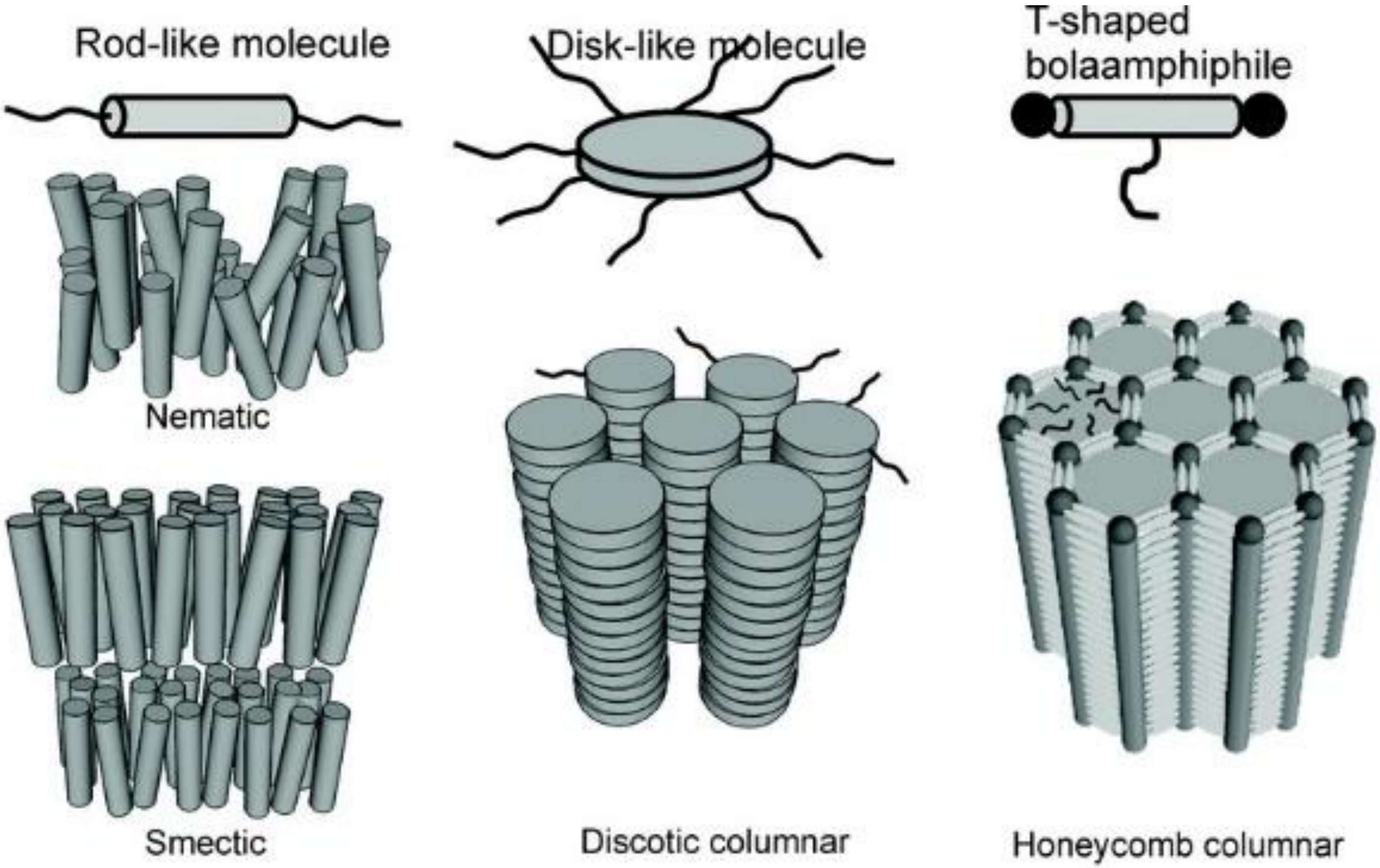
Concepts et propriétés physiques illustrés par des expériences

TOME 1

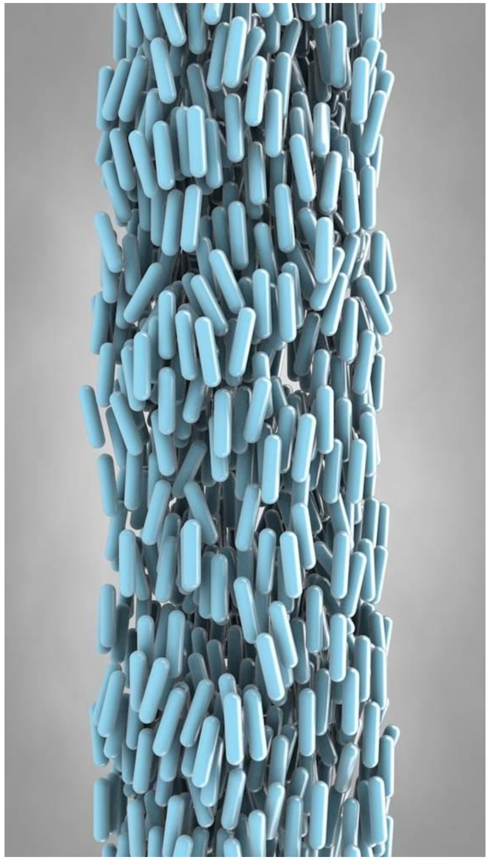
GB
Gordon and Breach Science Publishers

Appendices

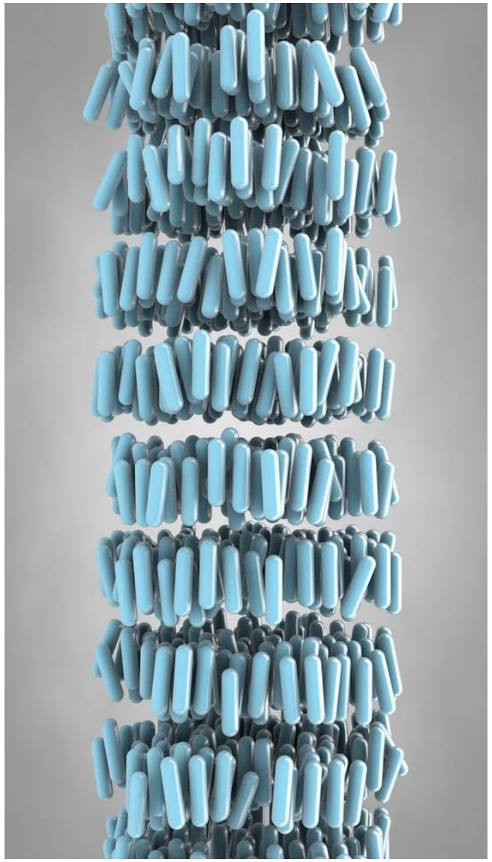
Other kinds of liquid crystals



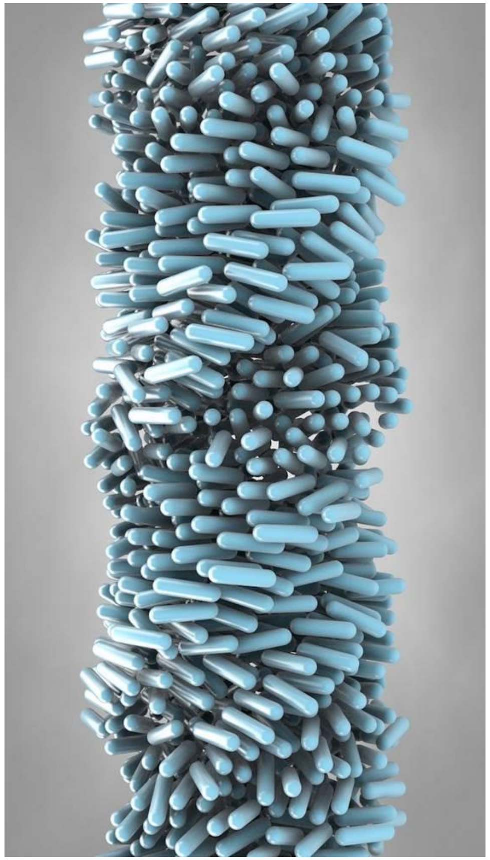
Other kinds of liquid crystals



Nematic



Smectic A

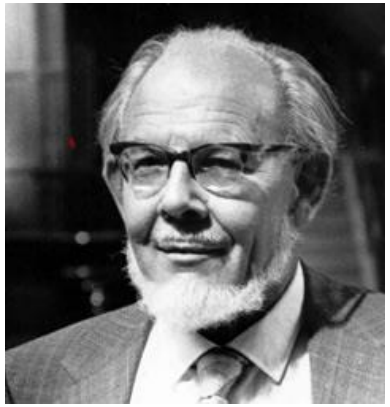


Cholesteric

Blue phases

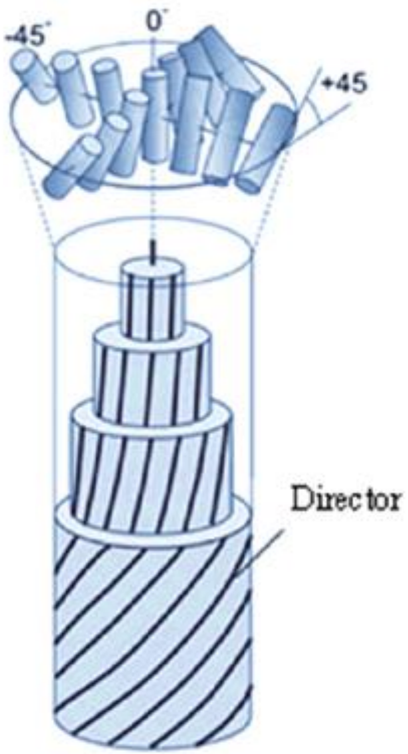
E Oton et al. Sci Rep 7 (2017)

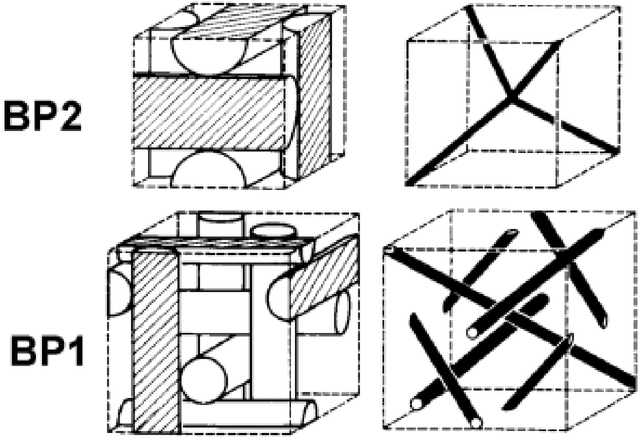
« They are totally useless, I think, except for one important intellectual use, that of providing tangible examples of topological oddities, and so helping to bring topology into the public domain of science »



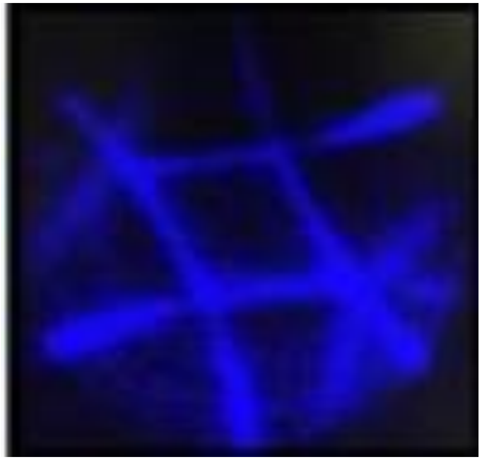
FC Frank

Recent topic (G Gray, 1956) because they only exist within a very narrow range of $\triangle T < 1°C$.





disclinations network



Kossel rings

Blue phases

E Oton et al. Sci Rep 7 (2017)

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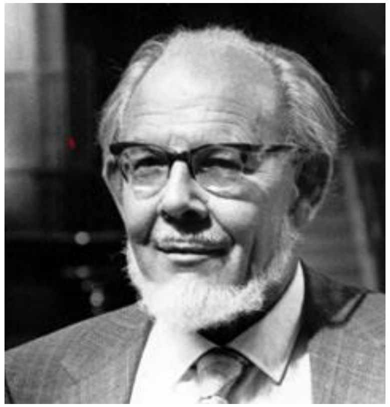
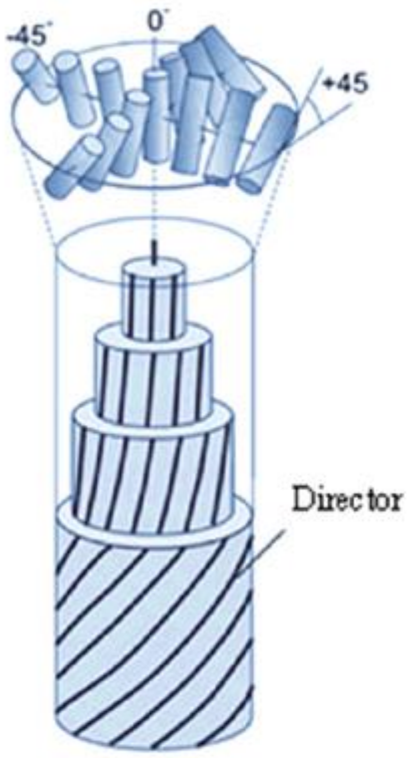
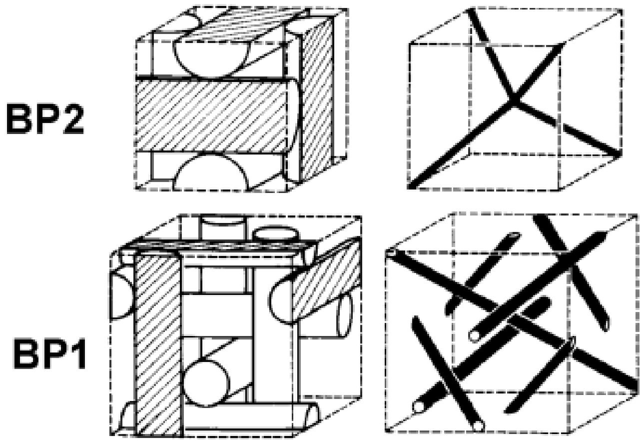


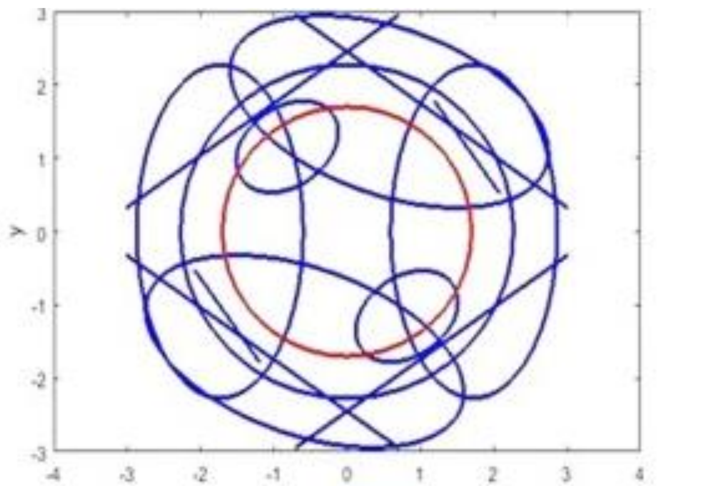
FC Frank

Recent topic (G Gray, 1956) because they only exist within a very narrow range of $\triangle T < 1°\text{C}$.



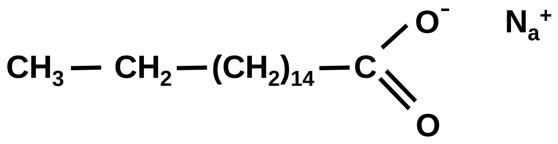


disclinations network

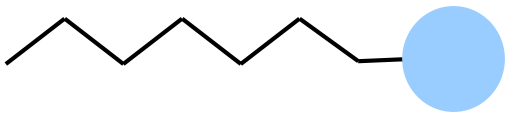


Kossel rings

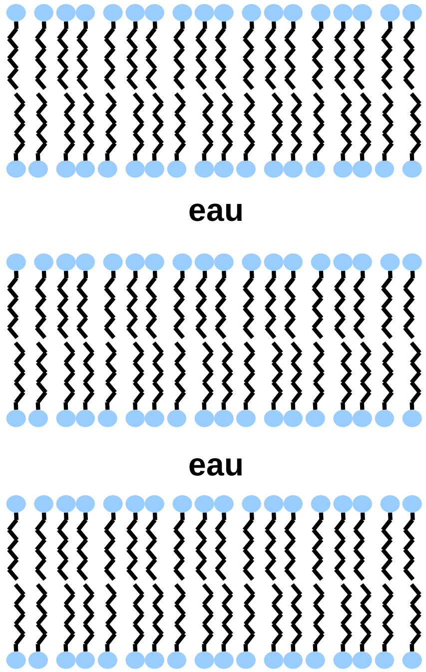
Lyotropic mesogens



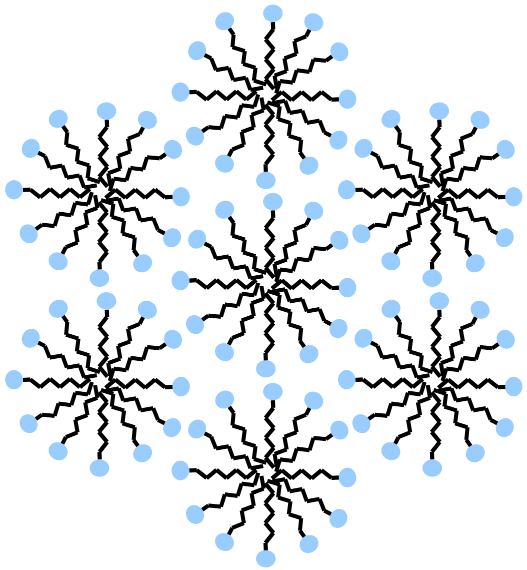
$$CH_3-CH_2-(CH_2)_{14}-C\overset{O^-}{\underset{O}{\lessgtr}}\quad Na^+$$



soap



smectic phase



hexagonal phase