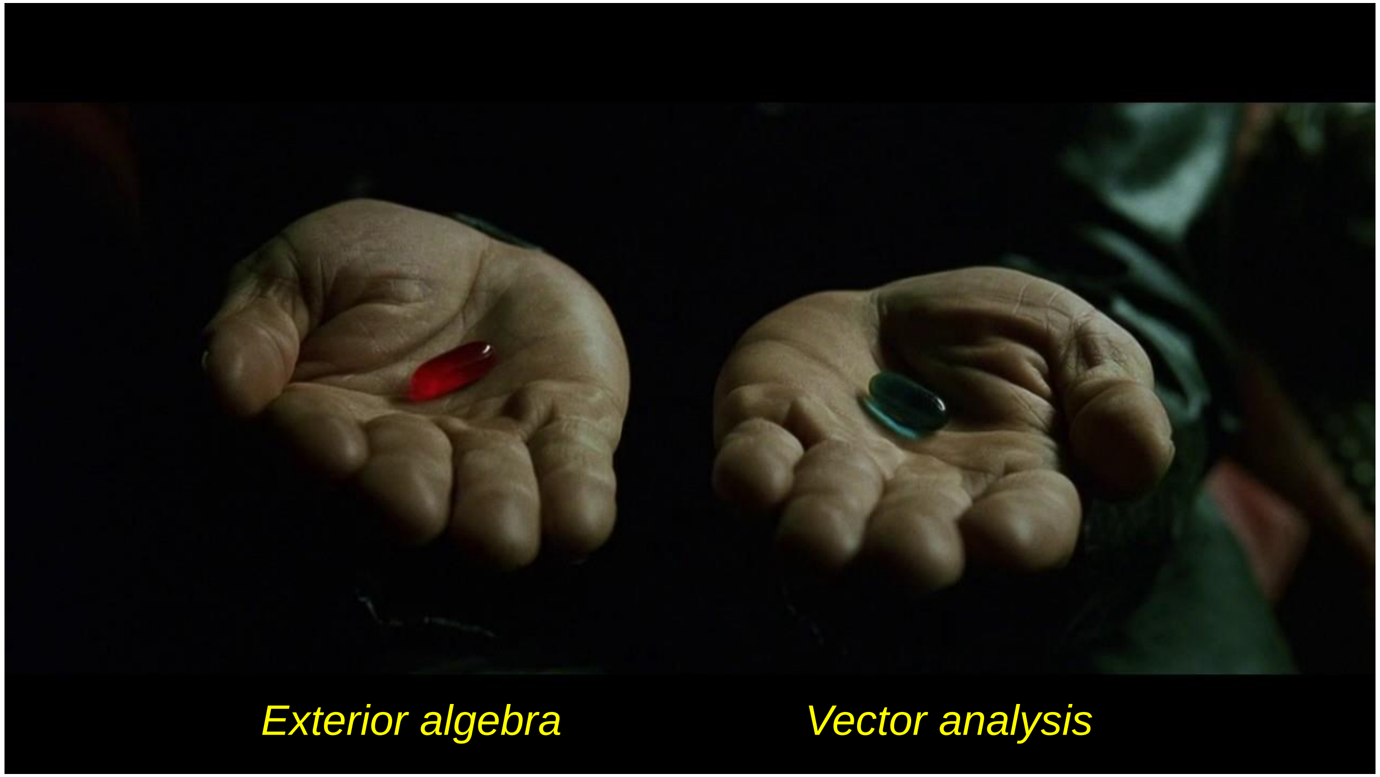
Sébastien Fumeron

IJL, April 2019

ELECTRODYNAMICS RELOADED

Differential forms, premetric theory and all that…



UNIVERSITÉ DE LORRAINE

Laboratoire LPCT Physique et Chimie Théoriques

Part I

« Forms illuminates electromagnetism »

A practical introduction to differential forms

$\wedge \quad d \quad \star \quad \delta$

- **Maxwell's equations**

$$\vec{\nabla}\times\vec{E} = -\partial_t\vec{B} \qquad \vec{\nabla}\times\vec{H} = \vec{j} + \partial_t\vec{D} \qquad \vec{D} = \varepsilon\vec{E}$$

$$\vec{\nabla}.\vec{D} = \rho \qquad \vec{\nabla}.\vec{B} = 0 \qquad \vec{B} = \mu\vec{H}$$

AJ Ward, JB Pendry. J. Mod. Opt. **43** 773 (1996)

- **Maxwell's equations**

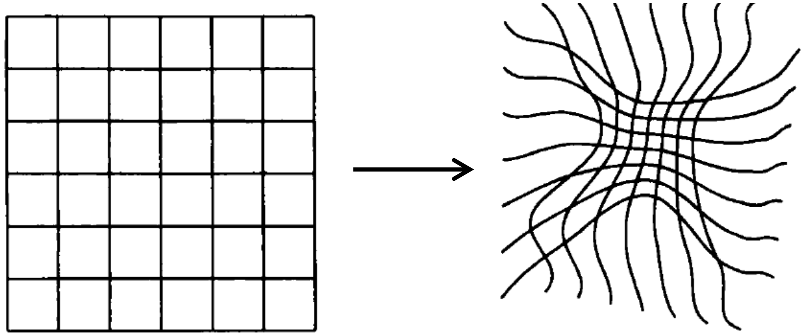
$$\vec{\nabla}\times\vec{E} = -\partial_t\vec{B} \qquad \vec{\nabla}\times\vec{H} = \vec{j} + \partial_t\vec{D} \qquad \vec{D} = \varepsilon\vec{E}$$

$$\vec{\nabla}.\vec{D} = \rho \qquad \vec{\nabla}.\vec{B} = 0 \qquad \vec{B} = \mu\vec{H}$$

↪ **General coordinate changes** $\vec{x} \to \vec{u}(\vec{x}) = \begin{pmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{pmatrix}$

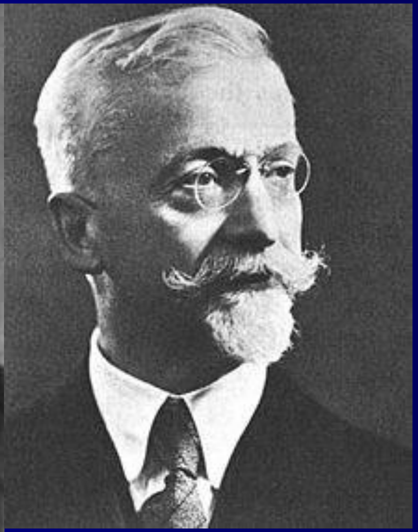
$$\vec{E}' = \overline{\overline{J}}^{-T}\vec{E} \qquad \vec{H}' = \overline{\overline{J}}^{-T}\vec{H}$$

$$\vec{D}' = \left(\frac{\overline{\overline{J}}}{\text{dét}\ \overline{\overline{J}}}\right)\vec{D} \qquad \vec{B}' = \left(\frac{\overline{\overline{J}}}{\text{dét}\ \overline{\overline{J}}}\right)\vec{B}$$



$\vec{x}, \vec{E}, \vec{B}$ $\longrightarrow$ $\vec{u}, \vec{E}', \vec{B}'$

1. Exterior calculus



Another look on total differentials

- **Euclidean space** $\mathbb{R}^3$ *coordinate system* $\left\{x^a\right\}_{a=1..3} = \left\{x, y, z\right\}$

$$T\left(x, y, z\right) \qquad dT = \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy + \frac{\partial T}{\partial z}dz = \frac{\partial T}{\partial x^a}dx^a \qquad a = 1, 2, 3$$

Another look on total differentials

- **Euclidean space** $\mathbb{R}^3$ *coordinate system* $\left\{x^a\right\}_{a=1..3} = \left\{x, y, z\right\}$

$$T\left(x, y, z\right) \qquad dT = \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy + \frac{\partial T}{\partial z}dz = \frac{\partial T}{\partial x^a}dx^a \qquad a = 1, 2, 3$$

$$V\left(x, y, z\right) \qquad dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = \frac{\partial V}{\partial x^a}dx^a \qquad a = 1, 2, 3$$

Another look on total differentials

- **Euclidean space** $\mathbb{R}^3$ *coordinate system* $\left\{x^a\right\}_{a=1..3} = \left\{x, y, z\right\}$

$$T\left(x,y,z\right) \qquad dT = \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy + \frac{\partial T}{\partial z}dz = \frac{\partial T}{\partial x^a}dx^a \qquad a = 1,2,3$$

$$V\left(x,y,z\right) \qquad dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = \frac{\partial V}{\partial x^a}dx^a \qquad a = 1,2,3$$

basis elements

Another look on total differentials

- **Euclidean space** $\mathbb{R}^3$ *coordinate system* $\left\{x^a\right\}_{a=1..3} = \left\{x, y, z\right\}$

$$T\left(x, y, z\right) \qquad dT = \frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial y} dy + \frac{\partial T}{\partial z} dz = \frac{\partial T}{\partial x^a} dx^a \qquad a = 1, 2, 3$$

$$V\left(x, y, z\right) \qquad dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz = \frac{\partial V}{\partial x^a} dx^a \qquad a = 1, 2, 3$$

basis elements

$$\alpha = u\left(x, y, z\right) dx + v\left(x, y, z\right) dy + w\left(x, y, z\right) dz = u_i\left(x, y, z\right) dx^i \in \Lambda^1\left(\mathbb{R}^3\right)$$

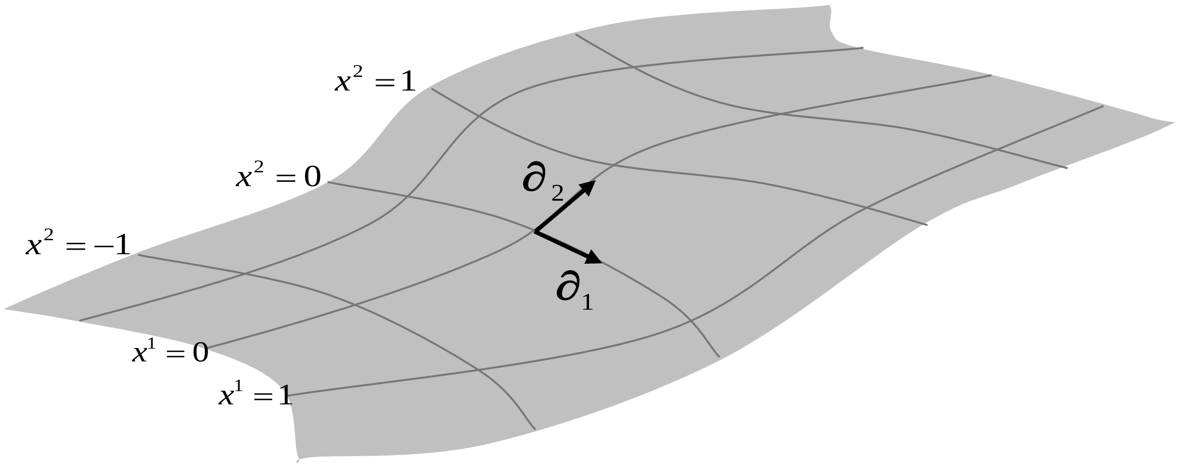
Forms on a manifold

- **n-manifold $\mathcal{M}$** = smooth hypersurface that locally « looks like » $\mathbb{R}^n$

coordinate system $\left\{x^a\right\}_{a=1..n}$

coordinate basis (tangent space) $\left\{e_a = \partial_a\right\}_{a=1..n}$

1-form basis (cotangent space) $\left\{dx^a\right\}_{a=1..n}$



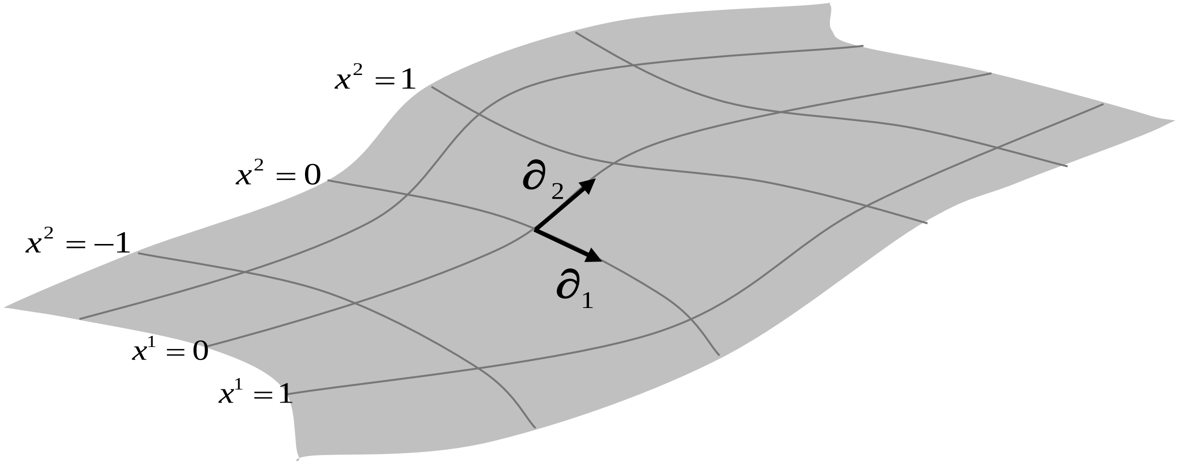
Forms on a manifold

- **n-manifold $\mathcal{M}$** = smooth hypersurface that locally « looks like » $\mathbb{R}^n$

coordinate system $\left\{x^a\right\}_{a=1..n}$

coordinate basis (tangent space) $\left\{e_a = \partial_a\right\}_{a=1..n}$

1-form basis (cotangent space) $\left\{dx^a\right\}_{a=1..n}$



$$dx^1 \otimes dx^2 - dx^2 \otimes dx^1 = dx^1 \wedge dx^2$$

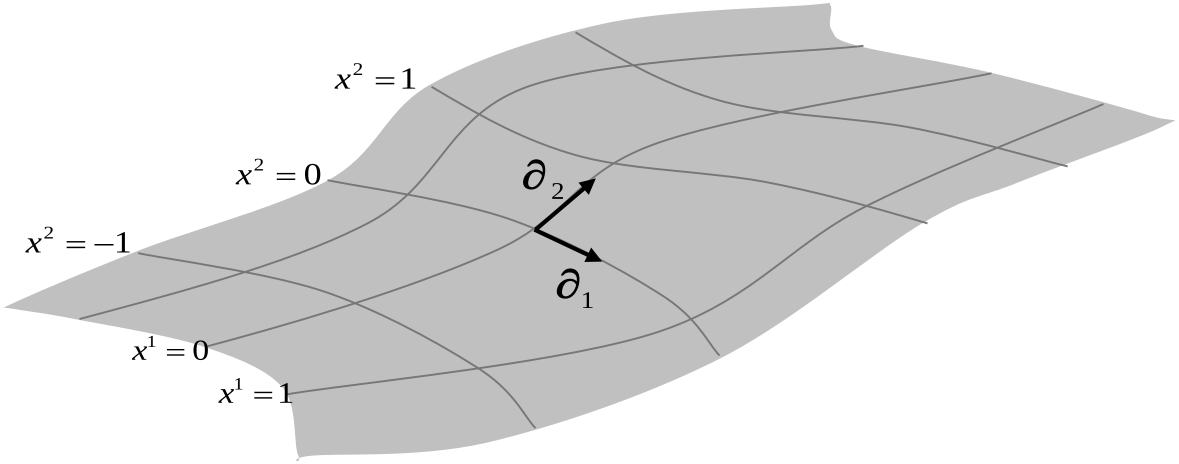
Forms on a manifold

- **n-manifold $\mathcal{M}$** = smooth hypersurface that locally « looks like » $\mathbb{R}^n$

coordinate system $\left\{x^a\right\}_{a=1..n}$

coordinate basis (tangent space) $\left\{e_a = \partial_a\right\}_{a=1..n}$

1-form basis (cotangent space) $\left\{dx^a\right\}_{a=1..n}$



$$dx^1 \otimes dx^2 - dx^2 \otimes dx^1 = dx^1 \wedge dx^2 \quad \Rightarrow \left\{dx^a \wedge dx^b, a \neq b\right\}_{a,b=1..n}$$

2-form basis

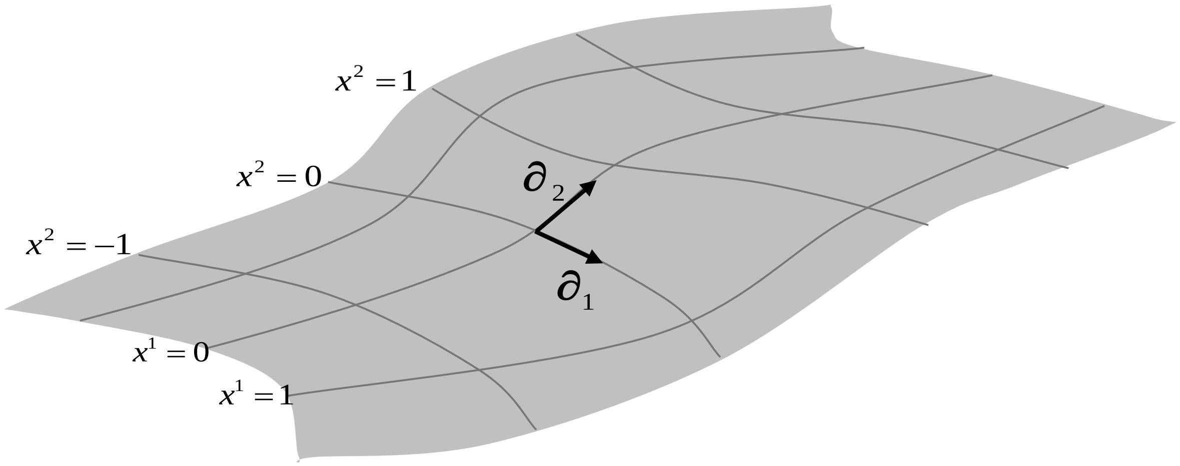
Forms on a manifold

- **n-manifold $\mathcal{M}$** = smooth hypersurface that locally « looks like » $\mathbb{R}^n$

coordinate system $\left\{x^a\right\}_{a=1..n}$

coordinate basis (tangent space) $\left\{e_a = \partial_a\right\}_{a=1..n}$

1-form basis (cotangent space) $\left\{dx^a\right\}_{a=1..n}$



$$dx^1 \otimes dx^2 - dx^2 \otimes dx^1 = dx^1 \wedge dx^2 \quad \Rightarrow \left\{dx^a \wedge dx^b, a \neq b\right\}_{a,b=1..n}$$

2-form basis

$$dx^1 \otimes dx^2 \otimes dx^3 + dx^3 \otimes dx^1 \otimes dx^2 + dx^2 \otimes dx^3 \otimes dx^1 - dx^1 \otimes dx^3 \otimes dx^2$$
$$-dx^3 \otimes dx^2 \otimes dx^1 - dx^2 \otimes dx^1 \otimes dx^3 = dx^1 \wedge dx^2 \wedge dx^3$$

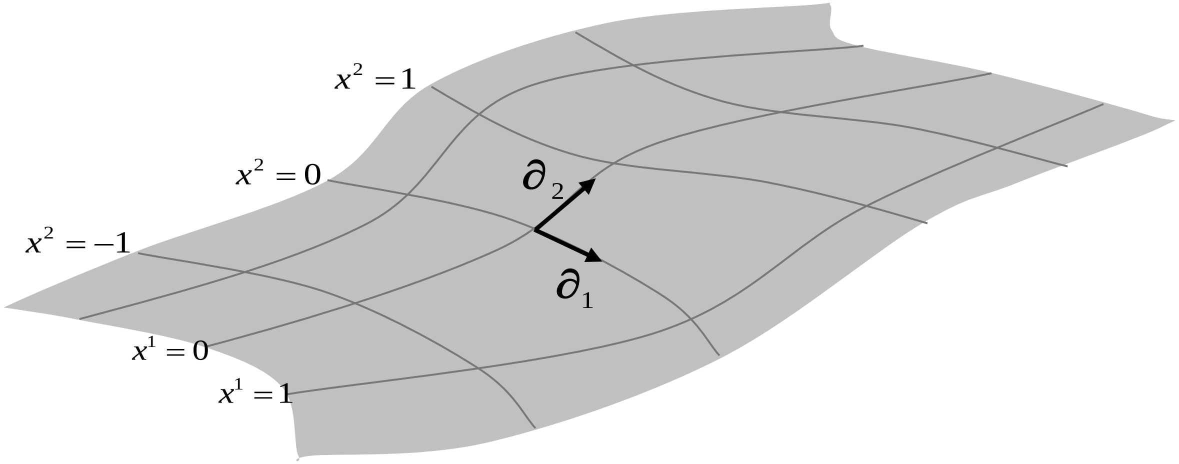
Forms on a manifold

- **n-manifold $\mathcal{M}$** = smooth hypersurface that locally « looks like » $\mathbb{R}^n$

coordinate system $\left\{x^a\right\}_{a=1..n}$

coordinate basis (tangent space) $\left\{e_a = \partial_a\right\}_{a=1..n}$

1-form basis (cotangent space) $\left\{dx^a\right\}_{a=1..n}$



$$dx^1 \otimes dx^2 - dx^2 \otimes dx^1 = dx^1 \wedge dx^2 \quad \Rightarrow \left\{dx^a \wedge dx^b, a \neq b\right\}_{a,b=1..n}$$

2-form basis

$$dx^1 \otimes dx^2 \otimes dx^3 + dx^3 \otimes dx^1 \otimes dx^2 + dx^2 \otimes dx^3 \otimes dx^1 - dx^1 \otimes dx^3 \otimes dx^2$$
$$-dx^3 \otimes dx^2 \otimes dx^1 - dx^2 \otimes dx^1 \otimes dx^3 = dx^1 \wedge dx^2 \wedge dx^3 \Rightarrow \left\{dx^a \wedge dx^b \wedge dx^c, a \neq b \neq c\right\}_{a,b,c=1..n}$$

3-form basis

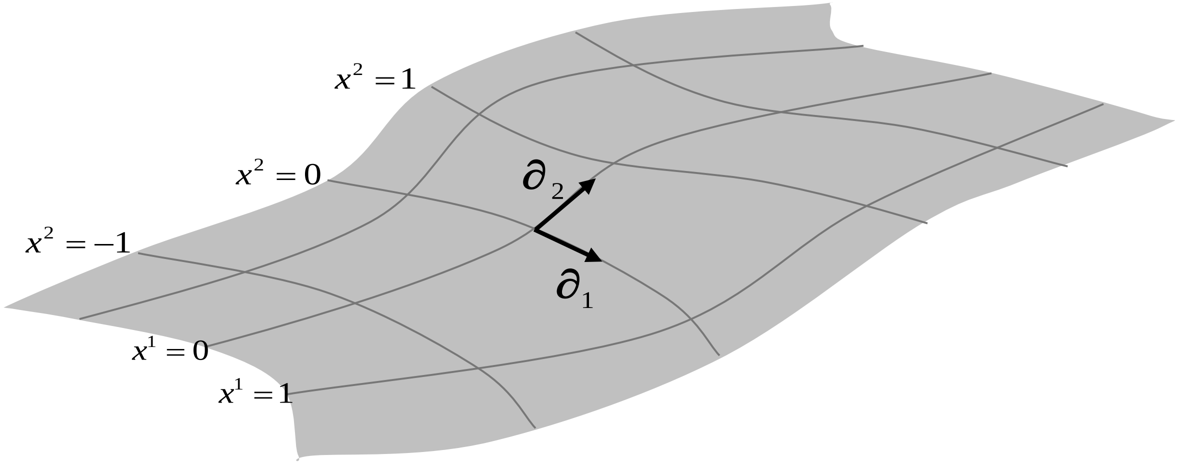
Forms on a manifold

- **n-manifold $\mathcal{M}$** = smooth hypersurface that locally « looks like » $\mathbb{R}^n$

coordinate system $\left\{x^a\right\}_{a=1..n}$

coordinate basis (tangent space) $\left\{e_a = \partial_a\right\}_{a=1..n}$

1-form basis (cotangent space) $\left\{dx^a\right\}_{a=1..n}$



- **Differential p-form ω** = fully anti-symmetric (0,p) tensor on $\mathcal{M}$

$$\omega = \omega_{a_1 \dots a_p} dx^{a_1} \wedge \dots \wedge dx^{a_p}$$

fully anti-symmetrized tensor product or wedge product

Forms of a given degree p are additive (as covectors) and define a vector space $\Lambda^p(\mathcal{M})$ of dimension :

$$C_n^p = \frac{n!}{p!(n-p)!}$$

Product of forms

- Given two forms u,v (degree m) and w (degree p), **the exterior product** $\wedge$ obeys the properties:

$$v \wedge w = (-1)^{mp}\, w \wedge v$$
$$u \wedge (v \wedge w) = (u \wedge v) \wedge w$$
$$(\lambda u + \mu v) \wedge w = \lambda u \wedge w + \mu v \wedge w$$

- Remarks :

1. $u \wedge w$ is a form of degree m+p
2. $dx^a \wedge dx^a = 0$ no form of degree p>n can exist
3. The set of all $\Lambda^p(\mathcal{M})$ p=0,..,n plus the wedge product define an algebra known as **Grassmann or exterior algebra.**

Wedge product in 3D

- The wedge product of two 1-forms :

$$u = u_x dx + u_y dy + u_z dz \qquad v = v_x dx + v_y dy + v_z dz$$

$$u \wedge v =$$

Wedge product in 3D

- The wedge product of two 1-forms :

$$u = u_x dx + u_y dy + u_z dz \qquad v = v_x dx + v_y dy + v_z dz$$

$$u \wedge v = \left(u_x v_y - u_y v_x\right) dx \wedge dy + \left(u_z v_x - u_x v_z\right) dz \wedge dx + \left(u_y v_z - u_z v_y\right) dy \wedge dz$$

Wedge product in 3D

- The wedge product of two 1-forms :

$$u = u_x dx + u_y dy + u_z dz \qquad v = v_x dx + v_y dy + v_z dz$$

$$u \wedge v = \left(u_x v_y - u_y v_x\right) dx \wedge dy + \left(u_z v_x - u_x v_z\right) dz \wedge dx + \left(u_y v_z - u_z v_y\right) dy \wedge dz$$

results in a 2-form with components equal to the "ordinary" cross product computed from the components of u and v.

Wedge product in 3D

- The wedge product of two 1-forms :

$$u = u_x dx + u_y dy + u_z dz \qquad v = v_x dx + v_y dy + v_z dz$$

$$u \wedge v = \left(u_x v_y - u_y v_x\right) dx \wedge dy + \left(u_z v_x - u_x v_z\right) dz \wedge dx + \left(u_y v_z - u_z v_y\right) dy \wedge dz$$

results in a 2-form with components equal to the "ordinary" cross product computed from the components of u and v.

- The wedge product of a 1-form u and a 2-form w :

$$u = u_1 dx + u_2 dy + u_3 dz \qquad w = w_1 dy \wedge dz + w_2 dz \wedge dx + w_3 dx \wedge dy$$

$$u \wedge w =$$

Wedge product in 3D

- The wedge product of two 1-forms :

$$u = u_x dx + u_y dy + u_z dz \qquad v = v_x dx + v_y dy + v_z dz$$

$$u \wedge v = \left(u_x v_y - u_y v_x\right) dx \wedge dy + \left(u_z v_x - u_x v_z\right) dz \wedge dx + \left(u_y v_z - u_z v_y\right) dy \wedge dz$$

results in a 2-form with components equal to the "ordinary" cross product computed from the components of u and v.

- The wedge product of a 1-form u and a 2-form w :

$$u = u_1 dx + u_2 dy + u_3 dz \qquad w = w_1 dy \wedge dz + w_2 dz \wedge dx + w_3 dx \wedge dy$$

$$u \wedge w = \left(u_1 w_1 + u_2 w_2 + u_3 w_3\right) dx \wedge dy \wedge dz$$

Wedge product in 3D

- The wedge product of two 1-forms :

$$u = u_x dx + u_y dy + u_z dz \qquad v = v_x dx + v_y dy + v_z dz$$

$$u \wedge v = \left(u_x v_y - u_y v_x\right) dx \wedge dy + \left(u_z v_x - u_x v_z\right) dz \wedge dx + \left(u_y v_z - u_z v_y\right) dy \wedge dz$$

results in a 2-form with components equal to the "ordinary" cross product computed from the components of u and v.

- The wedge product of a 1-form u and a 2-form w :

$$u = u_1 dx + u_2 dy + u_3 dz \qquad w = w_1 dy \wedge dz + w_2 dz \wedge dx + w_3 dx \wedge dy$$

$$u \wedge w = \left(u_1 w_1 + u_2 w_2 + u_3 w_3\right) dx \wedge dy \wedge dz$$

results in a 2-form with components equal to the "ordinary" scalar product of computed from the components of u and w.

Forms as integrands

"… the things which occur under integral signs."

H Flanders , Differential forms with applications to physical science (1989)

p=0 : $\omega = A =$ scalar

p=1 : $\omega = \omega_a dx^a \quad \Rightarrow$ integrate once : $\int_C \omega_a dx^a =$ scalar $\qquad \leftrightarrow \quad \langle bra|$

p=2 : $\omega = \omega_{a_1 a_2} dx^{a_1} \wedge dx^{a_2} = \omega_{a_1 a_2}\left(dx^{a_1} \otimes dx^{a_2} - dx^{a_2} \otimes dx^{a_1}\right)$

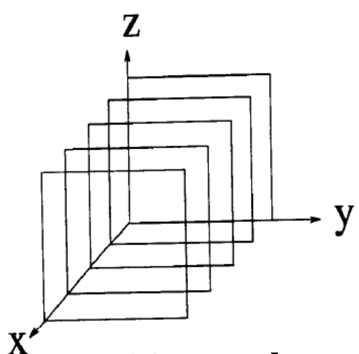
$\Rightarrow$ integrate twice : $\iint_S \omega_{a_1 a_2} dx^{a_1} \wedge dx^{a_2} =$ scalar

p $\leq n \quad \Rightarrow$ integrate p-times : $\iint .. \int_{D^p} \omega_{a_1 \ldots a_p} dx^{a_1} \wedge \ldots \wedge dx^{a_p} =$ scalar

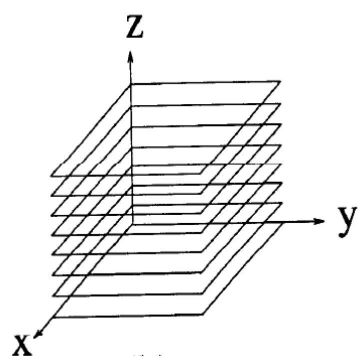
A differential p-form is an object you need to integrate p-times to get a scalar.

Picturing forms

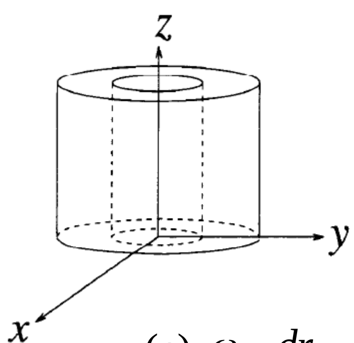
1-forms ↔ (curved) surfaces



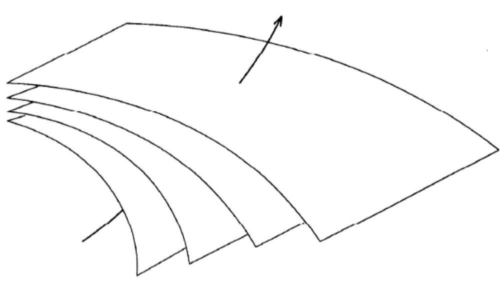
(a) $\omega = dx$



(b) $\omega = 2dz$



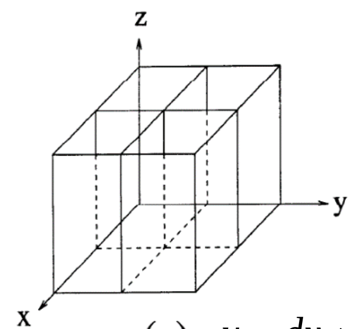
(c) $\omega = dr$



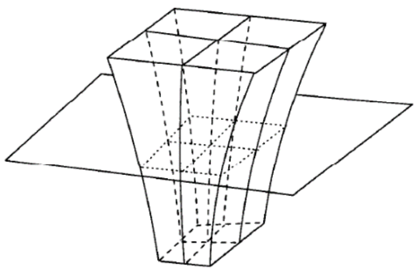
$\omega = \omega_a dx^a$

density of surfaces $x^a = C^{st}$
per unit length

2-forms ↔ (curved) tubes

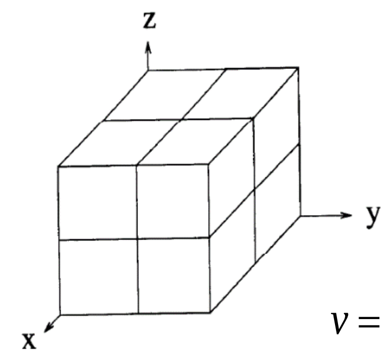


(a) $u = dx \wedge dy$



(b) $u = u_{a_1 a_2} dx^{a_1} \wedge dx^{a_2}$

3-forms ↔ (curved) boxes



$v = dx \wedge dy \wedge dz$

Definition and properties

- **The exterior derivative** is a linear operator denoted as d and in dimension n, it is defined formally as:

$$a = 1,..n \qquad \boxed{d. = \left(\frac{\partial}{\partial x^a} dx^a\right) \wedge .}$$

Unlike the ordinary derivative, it is dimensionless.

- **The exterior derivative** obeys the two following properties

$$\boxed{\begin{aligned} & d(a \wedge b) = (da) \wedge b + (-1)^p a \wedge db \\ & d(da) = 0 \end{aligned}}$$

Leibniz formula $p = \deg(a)$

Nilpotence

$$\Rightarrow d\left(f \; dx^{a_1} \wedge .. \wedge dx^{a_p}\right) = (df) \wedge dx^{a_1} \wedge .. \wedge dx^{a_p}$$

Action on p-forms in 3D

- The exterior derivative of a 0-form $f(x,y,z)$:

$$df = \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy + \frac{\partial}{\partial z} dz \right) \wedge f(x,y,z) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = \vec{\nabla} f . d\vec{r}$$

results in a 1-form with components equal to the "ordinary" gradient operator.

sidenote: consistency of notations (see $f(x,y,z) = x \Rightarrow d(f) = d(x) = dx$)

Action on p-forms in 3D

- The exterior derivative of a 0-form $f(x,y,z)$:

$$df=\left(\frac{\partial}{\partial x}dx+\frac{\partial}{\partial y}dy+\frac{\partial}{\partial z}dz\right)\wedge f(x,y,z)=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz=\vec{\nabla}f.d\vec{r}$$

results in a 1-form with components equal to the “ordinary” gradient operator.

sidenote: consistency of notations (see $f(x,y,z)=x\Rightarrow d(f)=d(x)=dx$)

- The exterior derivative of a 1-form $a=a_x dx+a_y dy+a_z dz$:

$$da=da_x\wedge dx+da_y\wedge dy+da_z\wedge dz=\left(\partial_y a_x\ dy\wedge dx+\partial_z a_x\ dz\wedge dx\right)+\ldots$$
$$\ldots+\left(\partial_x a_y\ dx\wedge dy+\partial_z a_y\ dz\wedge dy\right)+\left(\partial_x a_z\ dx\wedge dz+\partial_y a_z\ dy\wedge dz\right)$$

Action on p-forms in 3D

$$da = \left(\partial_x a_y - \partial_y a_x\right) dx \wedge dy + \left(\partial_y a_z - \partial_z a_y\right) dy \wedge dz + \left(\partial_z a_x - \partial_x a_z\right) dz \wedge dx$$

$$= \left(\vec{\nabla} \times \vec{a}\right) . d^2 \vec{S}$$

The exterior derivative of a 1-form results in a 2-form with components equal to the "ordinary" curl operator.

Action on p-forms in 3D

$$da = \left(\partial_x a_y - \partial_y a_x\right) dx \wedge dy + \left(\partial_y a_z - \partial_z a_y\right) dy \wedge dz + \left(\partial_z a_x - \partial_x a_z\right) dz \wedge dx$$
$$= \left(\vec{\nabla} \times \vec{a}\right) . d^2\vec{S}$$

The exterior derivative of a 1-form results in a 2-form with components equal to the "ordinary" curl operator.

- Similarly, the exterior derivative of a 2-form results in a 3-form with a component corresponding to the "ordinary" divergence operator.

$$b = b_x dy \wedge dz + b_y dz \wedge dx + b_z dx \wedge dy$$

$$db = \left(\partial_x b_x + \partial_y b_y + \partial_z b_z\right) dx \wedge dy \wedge dz = \left(\vec{\nabla} . \vec{b}\right) d^3V$$

How to use nilpotence and Leibniz formula ?

- Let f be a 0-form, $d\left(df\right)=0$

How to use nilpotence and Leibniz formula ?

- Let f be a 0-form, $\overset{\text{curl}}{d}\big(\underbrace{\overset{\text{grad}}{d}f}_{\text{1-form}}\big) = 0$

How to use nilpotence and Leibniz formula ?

- Let f be a 0-form, $\overset{\text{curl}}{d}\Big(\underbrace{\overset{\text{grad}}{d}f}_{\text{1-form}}\Big)=0$ $\Rightarrow$ translation : $\vec{\nabla}\times\left(\vec{\nabla}f\right)=\vec{0}$

How to use nilpotence and Leibniz formula ?

- Let f be a 0-form, $\overset{\text{curl}}{d}\Big(\underbrace{\overset{\text{grad}}{d}f}_{\text{1-form}}\Big) = 0 \ \Rightarrow$ translation : $\vec{\nabla} \times \left(\vec{\nabla} f\right) = \vec{0}$

- Let a be a 1-form, $d\left(da\right) = 0$

How to use nilpotence and Leibniz formula ?

- Let f be a 0-form, $\overset{\text{curl}}{d}\left(\underbrace{\overset{\text{grad}}{d}f}_{\text{1-form}}\right)=0 \Rightarrow$ translation : $\vec{\nabla}\times\left(\vec{\nabla}f\right)=\vec{0}$

- Let a be a 1-form, $\overset{\text{div}}{d}\left(\underbrace{\overset{\text{curl}}{d}a}_{\text{2-form}}\right)=0$

How to use nilpotence and Leibniz formula ?

- Let f be a 0-form, $\overset{\text{curl}}{d}\Big(\underbrace{\overset{\text{grad}}{d}f}_{\text{1-form}}\Big)=0$ $\Rightarrow$ translation : $\vec{\nabla}\times\left(\vec{\nabla}f\right)=\vec{0}$
- Let a be a 1-form, $\overset{\text{div}}{d}\Big(\underbrace{\overset{\text{curl}}{d}a}_{\text{2-form}}\Big)=0$ $\Rightarrow$ translation : $\vec{\nabla}.\left(\vec{\nabla}\times\vec{a}\right)=0$

How to use nilpotence and Leibniz formula ?

- Let f be a 0-form, $\overset{\text{curl}}{d}\underbrace{\left(\overset{\text{grad}}{d}f\right)}_{\text{1-form}}=0 \Rightarrow$ translation : $\vec{\nabla}\times\left(\vec{\nabla}f\right)=\vec{0}$
- Let a be a 1-form, $\overset{\text{div}}{d}\underbrace{\left(\overset{\text{curl}}{d}a\right)}_{\text{2-form}}=0 \Rightarrow$ translation : $\vec{\nabla}.\left(\vec{\nabla}\times\vec{a}\right)=0$
- Let f be a 0-form and a a 1-form, $d\left(f\wedge a\right)=\left(df\right)\wedge a+\left(-1\right)^{0} f\wedge da$

How to use nilpotence and Leibniz formula ?

- Let f be a 0-form, $\overset{\text{curl}}{d}\big(\underbrace{\overset{\text{grad}}{d}f}_{\text{1-form}}\big)=0 \Rightarrow$ translation : $\vec{\nabla}\times\left(\vec{\nabla}f\right)=\vec{0}$
- Let a be a 1-form, $\overset{\text{div}}{d}\big(\underbrace{\overset{\text{curl}}{d}a}_{\text{2-form}}\big)=0 \Rightarrow$ translation : $\vec{\nabla}.\left(\vec{\nabla}\times\vec{a}\right)=0$
- Let f be a 0-form and a a 1-form, $d\big(\underbrace{f\overset{\text{simple product}}{\wedge}a}_{\text{1-form}}\big)=\underbrace{(df)}_{\text{1-form}}\overset{\text{cross product}}{\wedge}\underbrace{a}_{\text{1-form}}+(-1)^{0}\underbrace{f}_{\text{0-form}}\overset{\text{simple product}}{\wedge}\underbrace{da}_{\text{2-form}}$

How to use nilpotence and Leibniz formula ?

- Let f be a 0-form, $\overset{\text{curl}}{d}\left(\underset{\text{1-form}}{\overset{\text{grad}}{d}f}\right)=0 \Rightarrow$ translation : $\vec{\nabla}\times\left(\vec{\nabla}f\right)=\vec{0}$

- Let a be a 1-form, $\overset{\text{div}}{d}\left(\underset{\text{2-form}}{\overset{\text{curl}}{d}a}\right)=0 \Rightarrow$ translation : $\vec{\nabla}.\left(\vec{\nabla}\times\vec{a}\right)=0$

- Let f be a 0-form and a a 1-form, $d\left(\underbrace{f\overset{\text{simple product}}{\wedge}a}_{\text{1-form}}\right)=\underset{\text{1-form}}{\left(df\right)}\overset{\text{cross product}}{\wedge}\underset{\text{1-form}}{a}+\left(-1\right)^{0}\underset{\text{0-form}}{f}\overset{\text{simple product}}{\wedge}\underset{\text{2-form}}{da}$

$\Rightarrow$ translation : $\vec{\nabla}\times\left(f\ \vec{a}\right)=\left(\vec{\nabla}f\right)\times\vec{a}+f\vec{\nabla}\times a$

How to use nilpotence and Leibniz formula ?

- Let f be a 0-form, $\overset{\text{curl}}{d}\big(\underbrace{\overset{\text{grad}}{d}f}_{\text{1-form}}\big)=0 \Rightarrow$ translation : $\vec{\nabla}\times\left(\vec{\nabla}f\right)=\vec{0}$

- Let a be a 1-form, $\overset{\text{div}}{d}\big(\underbrace{\overset{\text{curl}}{d}a}_{\text{2-form}}\big)=0 \Rightarrow$ translation : $\vec{\nabla}.\left(\vec{\nabla}\times\vec{a}\right)=0$

- Let f be a 0-form and a a 1-form, $d\big(\underbrace{f\overset{\text{simple product}}{\wedge}a}_{\text{1-form}}\big)=\big(\underbrace{df}_{\text{1-form}}\big)\overset{\text{cross product}}{\wedge}\underbrace{a}_{\text{1-form}}+(-1)^{0}\underbrace{f}_{\text{0-form}}\overset{\text{simple product}}{\wedge}\underbrace{da}_{\text{2-form}}$

  $\Rightarrow$ translation : $\vec{\nabla}\times\left(f\ \vec{a}\right)=\left(\vec{\nabla}f\right)\times\vec{a}+f\vec{\nabla}\times a$

- Let a and b be two1-forms, $d\left(a\wedge b\right)=\left(da\right)\wedge b+(-1)^{1}a\wedge db$

How to use nilpotence and Leibniz formula ?

- Let f be a 0-form, $\overset{\text{curl}}{d}\Big(\underbrace{\overset{\text{grad}}{d}f}_{\text{1-form}}\Big)=0 \;\Rightarrow$ translation : $\vec{\nabla}\times\left(\vec{\nabla}f\right)=\vec{0}$

- Let a be a 1-form, $\overset{\text{div}}{d}\Big(\underbrace{\overset{\text{curl}}{d}a}_{\text{2-form}}\Big)=0 \;\Rightarrow$ translation : $\vec{\nabla}.\left(\vec{\nabla}\times\vec{a}\right)=0$

- Let f be a 0-form and a a 1-form, $d\Big(\underbrace{f\overset{\text{simple product}}{\wedge}a}_{\text{1-form}}\Big)=\Big(\underbrace{df}_{\text{1-form}}\Big)\overset{\text{cross product}}{\wedge}\underbrace{a}_{\text{1-form}}+(-1)^{0}\underbrace{f}_{\text{0-form}}\overset{\text{simple product}}{\wedge}\underbrace{da}_{\text{2-form}}$

$\Rightarrow$ translation : $\vec{\nabla}\times\left(f\ \vec{a}\right)=\left(\vec{\nabla}f\right)\times\vec{a}+f\vec{\nabla}\times a$

- Let a and b be two1-forms, $d\Big(\underbrace{a\overset{\text{cross product}}{\wedge}b}_{\text{2-form}}\Big)=\Big(\underbrace{da}_{\text{2-form}}\Big)\overset{\text{scalar product}}{\wedge}\underbrace{b}_{\text{1-form}}+(-1)^{1}\underbrace{a}_{\text{1-form}}\overset{\text{scalar product}}{\wedge}\underbrace{db}_{\text{2-form}}$

How to use nilpotence and Leibniz formula ?

- Let f be a 0-form, $\overset{\text{curl}}{d}\left(\underset{\text{1-form}}{\overset{\text{grad}}{d}f}\right)=0 \;\Rightarrow$ translation : $\vec{\nabla}\times\left(\vec{\nabla}f\right)=\vec{0}$

- Let a be a 1-form, $\overset{\text{div}}{d}\left(\underset{\text{2-form}}{\overset{\text{curl}}{d}a}\right)=0 \;\Rightarrow$ translation : $\vec{\nabla}.\left(\vec{\nabla}\times\vec{a}\right)=0$

- Let f be a 0-form and a a 1-form, $d\left(\underbrace{f\overset{\text{simple product}}{\wedge}a}_{\text{1-form}}\right)=\underset{\text{1-form}}{\left(df\right)}\overset{\text{cross product}}{\wedge}\underset{\text{1-form}}{a}+\left(-1\right)^{0}\underset{\text{0-form}}{f}\overset{\text{simple product}}{\wedge}\underset{\text{2-form}}{da}$

$$\Rightarrow \text{translation :}\quad \vec{\nabla}\times\left(f\ \vec{a}\right)=\left(\vec{\nabla}f\right)\times\vec{a}+f\vec{\nabla}\times a$$

- Let a and b be two1-forms, $d\left(\underbrace{a\overset{\text{cross product}}{\wedge}b}_{\text{2-form}}\right)=\underset{\text{2-form}}{\left(da\right)}\overset{\text{scalar product}}{\wedge}\underset{\text{1-form}}{b}+\left(-1\right)^{1}\underset{\text{1-form}}{a}\overset{\text{scalar product}}{\wedge}\underset{\text{2-form}}{db}$

$$\Rightarrow \text{translation :}\ \vec{\nabla}.\left(\vec{a}\times\vec{b}\right)=\left(\vec{\nabla}\times\vec{a}\right).\,\vec{b}-\left(\vec{\nabla}\times\vec{b}\right).\,\vec{a}$$

Generalized Stokes' theorem

$$\boxed{\int_{\partial D} a = \int_D da}$$

$a =$ p-form

$\dim D = p+1$

Generalized Stokes' theorem

$$\boxed{\int_{\partial D} a = \int_D da}$$

$a =$ p-form

$\dim D = p+1$

$p=1$

$$a = a_x dx + a_y dy + a_z dz$$

$$\oint_{\partial D} \underbrace{a}_{\text{1-form}} = \int_D \overset{\text{curl}}{d}a$$

$$\Rightarrow \oint_C \vec{a}.d\vec{\ell} = \iint_{\Sigma_c} \left(\vec{\nabla} \times \vec{a}\right).d^2\vec{S}$$

Stokes formula

Generalized Stokes' theorem

$$\int_{\partial D} a = \int_D da$$

$a =$ p-form

$\dim D = p+1$

$p=1$

$$a = a_x dx + a_y dy + a_z dz$$

$$\oint_{\partial D} a = \int_D \overset{\text{curl}}{da}$$

1-form

$$\Rightarrow \oint_C \vec{a}.d\vec{\ell} = \iint_{\Sigma_c} \left(\vec{\nabla}\times\vec{a}\right).d^2\vec{S}$$

Stokes formula

$p=2$

$$b = b_x dy\wedge dz + b_y dz\wedge dx + b_z dx\wedge dy$$

$$\oint_{\partial D} b = \int_D \overset{\text{div}}{db}$$

2-form

$$\Rightarrow \oiint_{\Sigma} \vec{b}.\,d^2\vec{S} = \iiint_{V_\Sigma} \vec{\nabla}.\,\vec{b}\;d^3V$$

Green-Ostrogradski formula

Status of E

$\vec{\nabla} \times \vec{E} = -\partial_t \vec{B}$ $\Rightarrow$ translation : E is a 1-form

Status of E

$\vec{\nabla} \times \vec{E} = -\partial_t \vec{B}$ ⇒ translation : E is a 1-form

↪ **Coordinate change** $\vec{x} \to \vec{u}(\vec{x}) = \begin{pmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{pmatrix}$

Status of E

$\vec{\nabla}\times\vec{E} = -\partial_t \vec{B}$ ⇒ translation : E is a 1-form

↪ **Coordinate change** $\vec{x} \rightarrow \vec{u}(\vec{x}) = \begin{pmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{pmatrix}$ $\quad dx = \dfrac{\partial x}{\partial u}du + \dfrac{\partial x}{\partial v}dv + \dfrac{\partial x}{\partial w}dw$

Status of E

$\vec{\nabla}\times\vec{E} = -\partial_t \vec{B}$ ⇒ translation : E is a 1-form

↪ **Coordinate change** $\vec{x} \to \vec{u}(\vec{x}) = \begin{pmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{pmatrix}$ $dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv + \frac{\partial x}{\partial w} dw$

$$E = E_x dx + E_y dy + E_z dz = E_x \left(\frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv + \frac{\partial x}{\partial w} dw \right) + \ldots$$

Status of E

$\vec{\nabla}\times\vec{E} = -\partial_t \vec{B}$ ⇒ translation : E is a 1-form

↪ **Coordinate change** $\vec{x} \to \vec{u}(\vec{x}) = \begin{pmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{pmatrix}$ $\quad dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv + \frac{\partial x}{\partial w}dw$

$$E = E_x dx + E_y dy + E_z dz = E_x\left(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv + \frac{\partial x}{\partial w}dw\right) + \ldots = \left(\frac{\partial x}{\partial u}E_x + \frac{\partial y}{\partial u}E_y + \frac{\partial z}{\partial u}E_z\right)du + ..$$

Status of E

$\vec{\nabla}\times\vec{E}=-\partial_t\vec{B}$ ⇒ translation : E is a 1-form

↪ **Coordinate change** $\vec{x}\to\vec{u}(\vec{x})=\begin{pmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z)\end{pmatrix}$ $dx=\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv+\frac{\partial x}{\partial w}dw$

$$E=E_x dx+E_y dy+E_z dz=E_x\left(\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv+\frac{\partial x}{\partial w}dw\right)+\ldots=\left(\frac{\partial x}{\partial u}E_x+\frac{\partial y}{\partial u}E_y+\frac{\partial z}{\partial u}E_z\right)du+..$$

$$E_u=\left(\overline{\overline{J}}^{-T}\right)_{11}E_x+\left(\overline{\overline{J}}^{-T}\right)_{12}E_y+\left(\overline{\overline{J}}^{-T}\right)_{13}E_z$$

Status of E

$\vec{\nabla}\times\vec{E}=-\partial_t\vec{B}$ $\Rightarrow$ translation : E is a 1-form

↪ **Coordinate change** $\quad \vec{x}\to\vec{u}(\vec{x})=\begin{pmatrix} u(x,y,z)\\ v(x,y,z)\\ w(x,y,z)\end{pmatrix}$ $\quad dx=\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv+\frac{\partial x}{\partial w}dw$

$$E=E_x dx+E_y dy+E_z dz=E_x\left(\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv+\frac{\partial x}{\partial w}dw\right)+\ldots=\left(\frac{\partial x}{\partial u}E_x+\frac{\partial y}{\partial u}E_y+\frac{\partial z}{\partial u}E_z\right)du+..$$

$$E_u=\left(\overline{\overline{J}}^{-T}\right)_{11}E_x+\left(\overline{\overline{J}}^{-T}\right)_{12}E_y+\left(\overline{\overline{J}}^{-T}\right)_{13}E_z \quad\Rightarrow\begin{pmatrix}E_u\\E_v\\E_w\end{pmatrix}=\left(\overline{\overline{J}}^{-T}\right)\begin{pmatrix}E_x\\E_y\\E_z\end{pmatrix}$$

Status of B

$\vec{\nabla}.\vec{B}=0$ ⇒ translation : B is a 2-form

$$B=B_x dy\wedge dz+B_y dz\wedge dx+B_z dx\wedge dy$$

Status of B

$\vec{\nabla}.\vec{B}=0$ ⇒ translation : B is a 2-form $\quad B=B_x dy\wedge dz+B_y dz\wedge dx+B_z dx\wedge dy$

$$dy\wedge dz=\left(\frac{\partial y}{\partial u}du+\frac{\partial y}{\partial v}dv+\frac{\partial y}{\partial w}dw\right)\wedge\left(\frac{\partial z}{\partial u}du+\frac{\partial z}{\partial v}dv+\frac{\partial z}{\partial w}dw\right)$$

$$=\left(\frac{\partial y}{\partial v}\frac{\partial z}{\partial w}-\frac{\partial y}{\partial w}\frac{\partial z}{\partial v}\right)dv\wedge dw+\left(\frac{\partial y}{\partial w}\frac{\partial z}{\partial u}-\frac{\partial y}{\partial u}\frac{\partial z}{\partial w}\right)dw\wedge du+\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v}-\frac{\partial y}{\partial v}\frac{\partial z}{\partial u}\right)du\wedge dv$$

Status of B

$\vec{\nabla}.\vec{B}=0$ ⇒ translation : B is a 2-form

$$B = B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy$$

$$dy \wedge dz = \left(\frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv + \frac{\partial y}{\partial w} dw \right) \wedge \left(\frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv + \frac{\partial z}{\partial w} dw \right)$$

$$= \left(\frac{\partial y}{\partial v}\frac{\partial z}{\partial w} - \frac{\partial y}{\partial w}\frac{\partial z}{\partial v} \right) dv \wedge dw + \left(\frac{\partial y}{\partial w}\frac{\partial z}{\partial u} - \frac{\partial y}{\partial u}\frac{\partial z}{\partial w} \right) dw \wedge du + \left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\frac{\partial z}{\partial u} \right) du \wedge dv$$

$$dy \wedge dz = Com\left(\overline{\overline{J}}^{-1} \right)_{11} dv \wedge dw + Com\left(\overline{\overline{J}}^{-1} \right)_{12} dw \wedge du + Com\left(\overline{\overline{J}}^{-1} \right)_{13} du \wedge dv$$

Status of B

$$B = B_x \left[Com\left(\overline{\overline{J}}^{-1} \right)_{11} dv \wedge dw + Com\left(\overline{\overline{J}}^{-1} \right)_{12} dw \wedge du + Com\left(\overline{\overline{J}}^{-1} \right)_{13} du \wedge dv \right] + \ldots$$

Status of B

$$B = B_x \left[Com\left(\overline{\overline{J}}^{-1} \right)_{11} dv \wedge dw + Com\left(\overline{\overline{J}}^{-1} \right)_{12} dw \wedge du + Com\left(\overline{\overline{J}}^{-1} \right)_{13} du \wedge dv \right] + \ldots$$

$$= \left[B_x Com\left(\overline{\overline{J}}^{-1} \right)_{11} + B_y Com\left(\overline{\overline{J}}^{-1} \right)_{21} + B_z Com\left(\overline{\overline{J}}^{-1} \right)_{31} \right] dv \wedge dw + \ldots$$

Status of B

$$B = B_x \left[Com\left(\overline{\overline{J}}^{-1}\right)_{11} dv \wedge dw + Com\left(\overline{\overline{J}}^{-1}\right)_{12} dw \wedge du + Com\left(\overline{\overline{J}}^{-1}\right)_{13} du \wedge dv \right] + \ldots$$

$$= \left[B_x Com\left(\overline{\overline{J}}^{-1}\right)_{11} + B_y Com\left(\overline{\overline{J}}^{-1}\right)_{21} + B_z Com\left(\overline{\overline{J}}^{-1}\right)_{31} \right] dv \wedge dw + \ldots$$

$$= B_u dv \wedge dw + B_v dw \wedge du + B_w du \wedge dv$$

Status of B

$$B = B_x\left[Com\left(\overline{\overline{J}}^{-1}\right)_{11} dv\wedge dw + Com\left(\overline{\overline{J}}^{-1}\right)_{12} dw\wedge du + Com\left(\overline{\overline{J}}^{-1}\right)_{13} du\wedge dv\right] + \ldots$$

$$= \left[B_x Com\left(\overline{\overline{J}}^{-1}\right)_{11} + B_y Com\left(\overline{\overline{J}}^{-1}\right)_{21} + B_z Com\left(\overline{\overline{J}}^{-1}\right)_{31}\right] dv\wedge dw + \ldots$$

$$= B_u dv\wedge dw + B_v dw\wedge du + B_w du\wedge dv$$

$$\Rightarrow \begin{pmatrix} B_u \\ B_v \\ B_w \end{pmatrix} = Com\left(\overline{\overline{J}}^{-1}\right)^T \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \overline{\overline{J}} \left(\text{dét}\,\overline{\overline{J}}^{-1}\right) \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix} = \left(\frac{\overline{\overline{J}}}{\text{dét}\,\overline{\overline{J}}}\right) \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}$$

Equivalence table

EM quantity	Vector analysis	Exterior calculus
V	scalar field	0-form
$\vec{A}, \vec{E}, \vec{H}, \vec{M}$	vector field	1-form
$\vec{B}, \vec{D}, \vec{P}, \vec{j}$	vector field	2-form
ρ	scalar field	3-form

2. Hodge duality

Why ?

- In Maxwell's theory, there are only two fundamental fields, $\vec{E}$ and $\vec{B}$. $\vec{D}$ and $\vec{H}$ are simply auxiliary macroscopic fields (used only when matter is involved) as testified by the constitutive relations:

$$\vec{D} = \varepsilon \vec{E} \qquad\qquad \vec{B} = \mu \vec{H}$$

Why ?

- In Maxwell's theory, there are only two fundamental fields, $\vec{E}$ and $\vec{B}$. $\vec{D}$ and $\vec{H}$ are simply auxiliary macroscopic fields (used only when matter is involved) as testified by the constitutive relations:

$$\begin{array}{cc} \vec{D} = \varepsilon \vec{E} & \vec{B} = \mu \vec{H} \\ \updownarrow \quad \updownarrow & \updownarrow \quad \updownarrow \\ D \quad\ E & B \quad\ H \end{array}$$

2-form 1-form 2-form 1-form

Why ?

- In Maxwell's theory, there are only two fundamental fields, $\vec{E}$ and $\vec{B}$. $\vec{D}$ and $\vec{H}$ are simply auxiliary macroscopic fields (used only when matter is involved) as testified by the constitutive relations:

$$\begin{array}{ccc} \vec{D} & = \varepsilon & \vec{E} \\ \updownarrow & & \updownarrow \\ D & & E \\ \text{2-form} & & \text{1-form} \end{array} \qquad\qquad \begin{array}{ccc} \vec{B} & = \mu & \vec{H} \\ \updownarrow & & \updownarrow \\ B & & H \\ \text{2-form} & & \text{1-form} \end{array}$$

⇒ one can legitimately expect an operation that maps p-forms with (n-p)-forms and that inverts their parity.

⇒ Hodge duality

Additional ingredients

- **Metric structure on the n-manifold $\mathcal{M}$** = bilinear form defining the scalar product between vectors and determining lengths.

$$\boxed{g = g_{ab}\, dx^a \otimes dx^b}$$

metric tensor

$$g\left(\partial_a, \partial_b\right) = g_{ab}$$

$\Rightarrow$ Pythagoras theorem

- **The inner product <.,.> between two p-forms on $\mathcal{M}$** is defined as:
  - p=1: $\left\langle dx^a, dx^b \right\rangle = g^{ab}$
  - p=2: $\left\langle dx^{a_1} \wedge dx^{a_2}, dx^{b_1} \wedge dx^{b_2} \right\rangle = \begin{vmatrix} \left\langle dx^{a_1}, dx^{b_1} \right\rangle & \left\langle dx^{a_1}, dx^{b_2} \right\rangle \\ \left\langle dx^{a_2}, dx^{b_1} \right\rangle & \left\langle dx^{a_2}, dx^{b_2} \right\rangle \end{vmatrix}$

Additional ingredients

- **Metric structure on the n-manifold $\mathcal{M}$** = bilinear form defining the scalar product between vectors and determining lengths.

$$\boxed{g = g_{ab} dx^a \otimes dx^b}$$

metric tensor

$$g\left(\partial_a, \partial_b\right) = g_{ab}$$

⇒ Pythagoras theorem

- **The inner product <.,.> between two p-forms on $\mathcal{M}$** is defined as:
  - p=1: $\left\langle dx^a, dx^b \right\rangle = g^{ab}$
  - p>1: $$\left\langle dx^{a_1} \wedge .. \wedge dx^{a_p}, dx^{b_1} \wedge .. \wedge dx^{b_p} \right\rangle = \begin{vmatrix} \left\langle dx^{a_1}, dx^{b_1} \right\rangle & \cdots & \left\langle dx^{a_1}, dx^{b_p} \right\rangle \\ \vdots & \ddots & \vdots \\ \left\langle dx^{a_p}, dx^{b_1} \right\rangle & \cdots & \left\langle dx^{a_p}, dx^{b_p} \right\rangle \end{vmatrix}$$

Recovering length

- **The Hodge dual operator (or star operator) ★** is an invertible linear map between Λ^{n-p} and Λ^{p} such that:

$$\boxed{u \wedge (\star\, v) = \langle u, v\rangle \sqrt{|g|}\; dx^1 \wedge .. \wedge dx^n}$$

$$u, v \in \Lambda^p$$
$$g = \left|\det(g_{ab})\right|$$

$$\star(\star\, u) = (-1)^{s+p(n-p)}\, u \qquad u \wedge (\star\, v) = v \wedge (\star\, u)$$

$$\star\,(c_1 u + c_2 v) = c_1 (\star\, u) + c_2 (\star\, v) \qquad u \wedge (\star\, u) = 0 \Rightarrow u = 0$$

- Exemple : action on the n-form basis for flat space in spherical coordinates

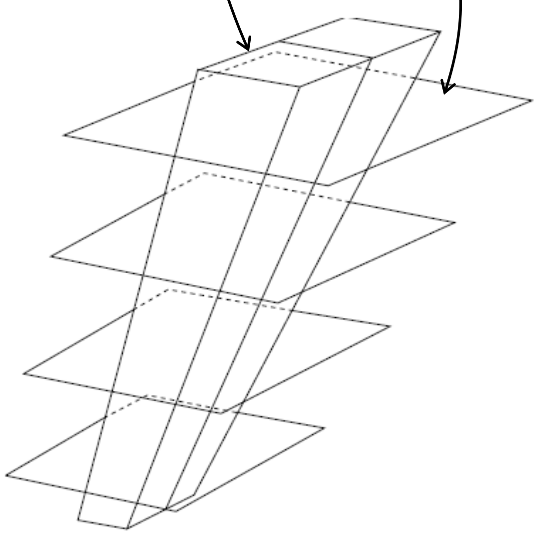
$$\star(d\theta \wedge d\varphi) = \frac{1}{r^2 \sin\theta}\, dr \qquad \star(dr \wedge d\theta) = \sin\theta\, d\varphi$$

$$\star(d\varphi \wedge dr) = \frac{1}{\sin\theta}\, d\theta \qquad \star(dr \wedge d\theta \wedge d\varphi) = \frac{1}{r^2 \sin\theta}$$

« Plane-to-tube transforms »

$$\vec{D} = \varepsilon \vec{E} \quad \updownarrow \quad D = \varepsilon \star E$$

2-form $D = \varepsilon \star E$ 1-form



$$\vec{B} = \mu \vec{H} \quad \updownarrow \quad B = \mu \star H$$

2-form 1-form

$$\vec{j} = \sigma \vec{E} \quad \updownarrow \quad j = \sigma \star E$$

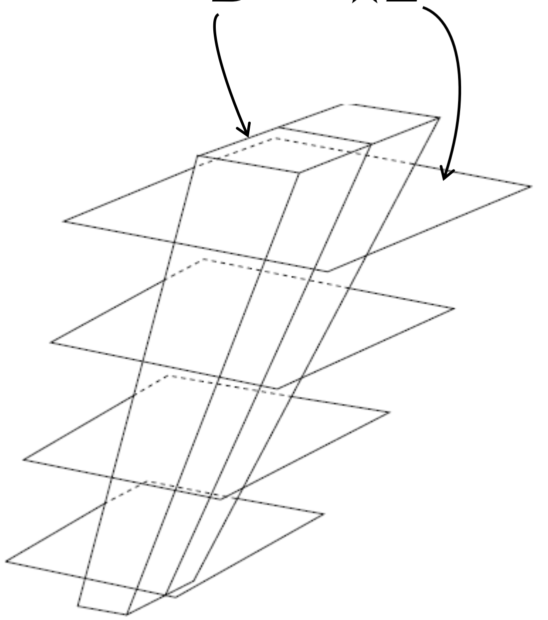
2-form 1-form

« Plane-to-tube transforms »

$$\vec{D} = \varepsilon \vec{E}$$

$$\updownarrow \quad \updownarrow$$

2-form $D = \varepsilon \star E$ 1-form



$$\vec{B} = \mu \vec{H}$$

$$\updownarrow \quad \updownarrow$$

$$B = \mu \star H$$

2-form 1-form

$$\vec{j} = \sigma \vec{E}$$

$$\updownarrow \quad \updownarrow$$

$$j = \sigma \star E$$

2-form 1-form

- Hodge operator is related to dual rotations of the EM field

$$\vec{E}' = \cos\alpha \vec{E} - \sin\alpha \vec{B}$$

$$\vec{B}' = \sin\alpha \vec{E} + \cos\alpha \vec{B}$$

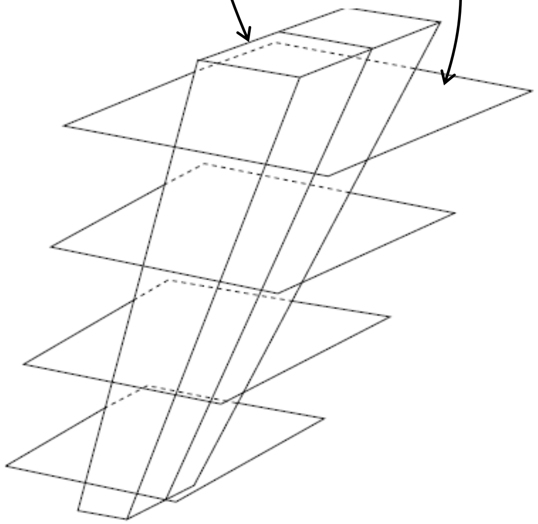
« Plane-to-tube transforms »

K Warnick, D Arnold. J. EM waves and ap. **16** (2012)

$$\vec{D} = \varepsilon \vec{E}$$

$$\updownarrow \quad \updownarrow$$

2-form $D = \varepsilon \star E$ 1-form



$$\vec{B} = \mu \vec{H}$$

$$\updownarrow \quad \updownarrow$$

$$B = \mu \star H$$

2-form 1-form

$$\vec{j} = \sigma \vec{E}$$

$$\updownarrow \quad \updownarrow$$

$$j = \sigma \star E$$

2-form 1-form

- Hodge operator is related to dual rotations of the EM field

$$\vec{E}' = \cos\alpha \vec{E} - \sin\alpha \vec{B}$$
$$\vec{B}' = \sin\alpha \vec{E} + \cos\alpha \vec{B}$$

- These relations can be extended to anisotropic media*.

3+1

Homogeneous equations

$$\vec{\nabla}\times\vec{E} = -\partial_t \vec{B}$$
$$\vec{\nabla}.\vec{B} = 0$$

$$dE = -\partial_t B$$
$$dB = 0$$

Inhomogeneous equations

$$\vec{\nabla}\times\vec{H} = \vec{j} + \partial_t \vec{D}$$
$$\vec{\nabla}.\vec{D} = \rho$$

$$dH = j + \partial_t D$$
$$dD = \rho$$

Constitutive relations

$$\vec{D} = \varepsilon\vec{E}$$
$$\vec{B} = \mu\vec{H}$$

$$D = \varepsilon \star E$$
$$B = \mu \star H$$

4D

- **Faraday 2-form:** $\boxed{F = E \wedge dt + B}$

4D

- **Faraday 2-form:** $\boxed{F = E \wedge dt + B}$

$$d_4 F = d_4 E \wedge dt + d_4 B$$

4D

- **Faraday 2-form:** $\boxed{F = E \wedge dt + B}$

$$d_4 F = d_4 E \wedge dt + d_4 B = d_3 E \wedge dt + \cancel{d_3 B} + (\partial_t B) \wedge dt = (-\partial_t B) \wedge dt + (\partial_t B) \wedge dt = 0$$

4D

- **Faraday 2-form:** $\boxed{F = E \wedge dt + B}$

$$d_4 F = d_4 E \wedge dt + d_4 B = d_3 E \wedge dt + \cancel{d_3 B} + \left(\partial_t B\right) \wedge dt = \left(-\partial_t B\right) \wedge dt + \left(\partial_t B\right) \wedge dt = 0$$

- **Maxwell 2-form:** $\boxed{G = D - H \wedge dt}$

$$\left(\varepsilon_0 = \mu_0 = 1\right)$$

4D

- **Faraday 2-form:** $\boxed{F = E \wedge dt + B}$

$$d_4 F = d_4 E \wedge dt + d_4 B = d_3 E \wedge dt + \cancel{d_3 B} + \left(\partial_t B\right) \wedge dt = \left(-\partial_t B\right) \wedge dt + \left(\partial_t B\right) \wedge dt = 0$$

- **Maxwell 2-form:** $\boxed{G = D - H \wedge dt}$ $\left(\varepsilon_0 = \mu_0 = 1\right)$

$$d_4 G = d_4 D - d_4 H \wedge dt = d_3 D + \left(\partial_t D\right) \wedge dt - d_3 H \wedge dt = \rho + \left(\partial_t D\right) \wedge dt - \left(j + \partial_t D\right) \wedge dt$$

$$= \rho - j \wedge dt = J$$

4D

- **Faraday 2-form:** $\boxed{F = E \wedge dt + B}$

$$d_4 F = d_4 E \wedge dt + d_4 B = d_3 E \wedge dt + \cancel{d_3 B} + (\partial_t B) \wedge dt = (-\partial_t B) \wedge dt + (\partial_t B) \wedge dt = 0$$

- **Maxwell 2-form:** $\boxed{G = D - H \wedge dt}$ $(\varepsilon_0 = \mu_0 = 1)$

$$d_4 G = d_4 D - d_4 H \wedge dt = d_3 D + (\partial_t D) \wedge dt - d_3 H \wedge dt = \rho + (\partial_t D) \wedge dt - (j + \partial_t D) \wedge dt$$

$$= \rho - j \wedge dt = J$$ $\Rightarrow$ charge conservation $d_4 J = 0$

4D

- **Faraday 2-form:** $\boxed{F = E \wedge dt + B}$

$$d_4 F = d_4 E \wedge dt + d_4 B = d_3 E \wedge dt + \cancel{d_3 B} + \left(\partial_t B\right) \wedge dt = \left(-\partial_t B\right) \wedge dt + \left(\partial_t B\right) \wedge dt = 0$$

- **Maxwell 2-form:** $\boxed{G = D - H \wedge dt}$ $\quad\left(\varepsilon_0 = \mu_0 = 1\right)$

$$d_4 G = d_4 D - d_4 H \wedge dt = d_3 D + \left(\partial_t D\right) \wedge dt - d_3 H \wedge dt = \rho + \left(\partial_t D\right) \wedge dt - \left(j + \partial_t D\right) \wedge dt$$

$$= \rho - j \wedge dt = J$$ $\Rightarrow$ charge conservation $d_4 J = 0$

- **Constitutive relation** $G = \star_4 F$

$\Rightarrow$ 4D formulation :

$$\boxed{\begin{aligned} d_4 F &= 0 \\ d_4 \star_4 F &= J \end{aligned}}$$

Topological (metric-free) equation

Metric-dependent equation

Summary

$$dE = -\partial_t B$$
$$dB = 0$$

metric-free structure equations

$$dH = j + \partial_t D$$
$$dD = \rho$$

metric-free source equations

$$D = \varepsilon \star E$$
$$B = \mu \star H$$

metric-coupled dual relations

Summary

$$dE = -\partial_t B$$
$$dB = 0$$

metric-free structure equations

$$dH = j + \partial_t D$$
$$dD = \rho$$

metric-free source equations

$$D = \varepsilon \star E$$
$$B = \mu \star H$$

metric-coupled dual relations

- **Vector analysis**

  $\vec{E}$ and $\vec{B}$ are two fundamental vectors.

  $\vec{D}$ and $\vec{H}$ are two auxilliary vectors relevant in matter only.

Exterior calculus

E and H are two 1-forms, D and B are two 2-forms.

Both pairs are dual quantities that have different physical meanings even in vacuum.

Summary

$$dE = -\partial_t B \qquad dH = j + \partial_t D \qquad D = \varepsilon \star E$$

$$dB = 0 \qquad dD = \rho \qquad B = \mu \star H$$

metric-free structure equations | *metric-free source equations* | *metric-coupled dual relations*

- **Vector analysis**

  $\vec{E}$ and $\vec{B}$ are two fundamental vectors.

  $\vec{D}$ and $\vec{H}$ are two auxilliary vectors relevant in matter only.

Exterior calculus

E and H are two 1-forms, D and B are two 2-forms.

Both pairs are dual quantities that have different physical meanings even in vacuum.

835.] MOLECULAR CURRENTS. 473

hollowed out of the substance of the magnet, Art. 395. We were thus led to consider two different quantities, the magnetic force and the magnetic induction, both of which are supposed to be observed in a space from which the magnetic matter is removed. We were not supposed to be able to penetrate into

Summary

$$dE = -\partial_t B \qquad dH = j + \partial_t D \qquad D = \varepsilon \star E$$

$$dB = 0 \qquad dD = \rho \qquad B = \mu \star H$$

metric-free structure equations *metric-free source equations* *metric-coupled dual relations*

- **Vector analysis**

  $\vec{E}$ and $\vec{B}$ are two fundamental vectors.

  $\vec{D}$ and $\vec{H}$ are two auxilliary vectors relevant in matter only.

  Exterior calculus

  E and H are two 1-forms, D and B are two 2-forms.

  Both pairs are dual quantities that have different physical meanings even in vacuum.

- All formula from vector analysis can be summed up/unified within a set of 3 equations: the nilpotence of d, Leibniz formula and the generalized integration formula. Exterior calculus extends them to n>3 dimensions.

Summary

VE Tarasov, Physics of plasmas, **13** 052107(2006)

$$dE = -\partial_t B$$
$$dB = 0$$

metric-free structure equations

$$dH = j + \partial_t D$$
$$dD = \rho$$

metric-free source equations

$$D = \varepsilon \star E$$
$$B = \mu \star H$$

metric-coupled dual relations

- **Vector analysis**

  $\vec{E}$ and $\vec{B}$ are two fundamental vectors.

  $\vec{D}$ and $\vec{H}$ are two auxilliary vectors relevant in matter only.

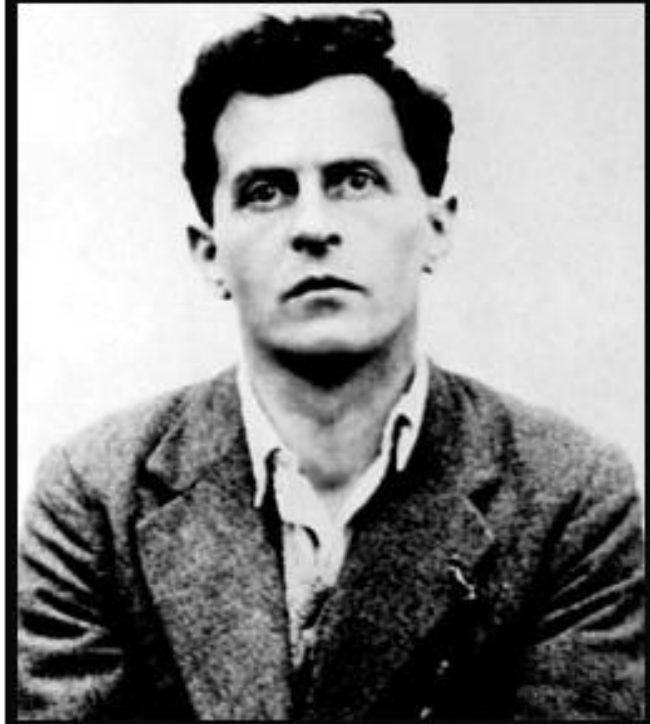
  Exterior calculus

  E and H are two 1-forms, D and B are two 2-forms.

  Both pairs are dual quantities that have different physical meanings even in vacuum.

- All formula from vector analysis can be summed up/unified within a set of 3 equations: the nilpotence of d, Leibniz formula and the generalized integration formula. Exterior calculus extends them to n>3 dimensions.

- Going further: fractional electrodynamics



If we spoke a different language, we would perceive a somewhat different world.

(Ludwig Wittgenstein)

[redacted]

Warning

The differential dx^i is an element of the cotangent space but not a tiny change.

Classical differential geometers (and classical analysts) did not hesitate to talk about "infinitely small" changes dx^i of the coordinates x^i, just as Leibnitz had. No one wanted to admit that this was nonsense, because true results were obtained when these infinitely small quantities were divided into each other (provided one did it in the right way).

Eventually it was realized that the closest one can come to describing an infinitely small change is to describe a direction in which this change is supposed to occur, i.e., a tangent vector. Since df is supposed to be the infinitesimal change of f under an infinitesimal change of the point, df must be a function of this change, which means that df should be a function on tangent vectors. The dx^i themselves then metamorphosed into functions, and it became clear that they must be distinguished from the tangent vectors $\partial/\partial x^i$.

M. Spivak. A comprehensive introduction to differential geometry, vol. 1 (1999)

De Rham cohomology

$\vec{B} = \vec{\nabla} \times \vec{A}$ **⇒ status of EM potentials and gauge invariance when using forms ?**

$B \overset{?}{=} dA$

- A p-form a is closed if $da = 0$. The set of closed p-forms is denoted as $\mathrm{Z}^p(\mathcal{M})$. A p-form *a* is exact if there is a p-1 form b such that $a = db$. The set of exact p-forms is denoted as $\mathrm{B}^p(\mathcal{M})$.

$$\mathrm{B}^p(\mathcal{M}) \subset \mathrm{Z}^p(\mathcal{M})$$

- Conversely, when is a closed p-form exact ?
  - *Locally, always (Poincaré's lemma).*
  - *Globally, it depends on the topology of $\mathcal{M}$. Defining De Rham p^{th} cohomology group:*

$$\mathrm{H}^p(\mathcal{M}) = \mathrm{Z}^p(\mathcal{M}) \,/\, \mathrm{B}^p(\mathcal{M})$$

dim $(\mathrm{H}^p(\mathcal{M})) = \mathrm{b}_p$ = p^{th} Betti number ↔ χ
= number of p-holes

↳ if $\mathrm{b}_p = 0$, then it is also exact.

↳ if $\mathrm{b}_p \neq 0$, then a closed p-form is exact if and only if all of its periods vanish (first De Rham theorem).

Electric and magnetic potentials

- On $\mathcal{M}=\mathbb{R}^3$, gauge symmetry is recovered from De Rham theorem for closed forms and nilpotence of d:
  - $dB=0 \Rightarrow$ De Rham theorem: $\exists$ a 1-form potential A such that $\boxed{B=dA}$
  - $dE=-\partial_t B=-\partial_t dA \Rightarrow d\left(E+\partial_t A\right)=0$

    $\Rightarrow$ De Rham theorem: $\exists$ a 0-form potential V such that $\boxed{E+\partial_t A=-dV}$
  - As $d^2=0$, $\begin{cases} B=dA \\ E+\partial_t A=-dV \end{cases}$ are invariant under $\begin{cases} A\to A+df \\ V\to V-\partial_t f \end{cases}$

The coderivative operator

↪ Nilpotence of *d* ⇒ How to build the wave equation ?

- **The coderivative operator δ** is defined in 3D as $\delta\omega = (-1)^p \star d \star \omega$ and it is a linear map from Λ^p to Λ^{p-1}. As for *d*, it is nilpotent.

- **The Laplace operator** is defined in 3D as $\Delta\omega = (d\delta + \delta d)\omega$ and it is a linear map from Λ^p to Λ^p. Its components are <u>opposite</u> to those of the usual Laplacian.

$$dE = -\partial_t B \Rightarrow \star dE = -\partial_t \star B = -\mu_0 \partial_t H \quad \Rightarrow d \star dE = -\mu_0 \partial_t dH$$

$$\Rightarrow d \star dE = -\mu_0 \partial_t \left(\varepsilon_0 \partial_t D\right) = -\frac{1}{c^2}\partial_t^2 D \quad \Rightarrow \quad \star d \star dE + \frac{1}{c^2}\partial_t^2 \star D = 0$$

$$\Rightarrow \delta dE + \frac{1}{c^2}\partial_t^2 E = 0$$

Wave equation, gauge, energy

Maxwell-Gauss $\Rightarrow 0 = dD = \varepsilon_0 d\star E \;\Rightarrow 0 = \delta E \Rightarrow 0 = d\delta E$

$$\Rightarrow \boxed{\Delta E + \frac{1}{c^2}\partial_t^2 E = 0}$$ D'Alembert equation

- Gauge condition $\Delta V = \delta dV + d\delta V = \delta\left(-E - \partial_t A\right) = -\partial_t\left(\delta A\right)$

$$\Rightarrow \Delta V + \frac{1}{c^2}\partial_t^2 V + \partial_t\left(-\frac{1}{c^2}\partial_t V + \delta A\right) = 0$$

$$\boxed{-\frac{1}{c^2}\partial_t V + \delta A = 0}$$ Lorenz gauge

- EM energy transport = Poynting theorem ⇒ S is a 2-form defined as $\boxed{S = E \wedge H}$

2-form 1-form 1-form

$$dS = d\left(E \wedge H\right) = dE \wedge H - E \wedge dH$$ (Leibniz formula)

$$= -\partial_t B \wedge H - E \wedge \partial_t D - E \wedge j = -\partial_t\left(\frac{1}{2}H \wedge B + \frac{1}{2}E \wedge D\right) - E \wedge j$$

Part II

Topological electromagnetism

A short introduction to premetric electrodynamics

Previously in ED reloaded

- **Compendium of exterior calculus:**

1) Differential p-form = fully anti-symmetric (0,p) tensor on a *n*-manifold $\mathcal{M}$

$$\omega = \omega_{a_1 \dots a_p} dx^{a_1} \wedge \dots \wedge dx^{a_p} \qquad \boxed{p \leq n}$$

A differential p-form is an object you need to integrate p-times to get a scalar.

2) Wedge product ∧ = anti-symmetrized tensor product (= scalar product, cross product)

3) Exterior derivative d (= grad, rot, div) $\quad d. = \left(\frac{\partial}{\partial x^a} dx^a \right) \wedge . \qquad d^2 \equiv 0$

4) Hodge star operator ★ = involves the metric

3+1 D

$$dE = -\partial_t B \qquad dH = j + \partial_t D \qquad D = \varepsilon \star E$$

$$dB = 0 \qquad dD = \rho \qquad B = \mu \star H$$

4 D

$$d_4 F = 0 \qquad d_4 G = J$$

$$G = \star_4 F$$

Axiomatics

- **Four Maxwell's equations ?**

Axiomatics

- **Four Maxwell's equations ?**

$$\vec{\nabla}.\vec{B}=0 \Rightarrow \partial_t\left(\vec{\nabla}.\vec{B}\right)=\vec{\nabla}.\left(\partial_t\vec{B}\right)=0 \Rightarrow \partial_t\vec{B} \text{ is a “pure curl” } \partial_t\vec{B}=\vec{\nabla}\times$$

Axiomatics

- **Four Maxwell's equations ?**

$$\vec{\nabla}.\vec{B}=0 \Rightarrow \partial_t\left(\vec{\nabla}.\vec{B}\right)=\vec{\nabla}.\left(\partial_t\vec{B}\right)=0 \Rightarrow \partial_t\vec{B} \text{ is a “pure curl” } \partial_t\vec{B}=\vec{\nabla}\times\left(-\vec{E}\right)$$

$$dB=0 \Rightarrow \partial_t\left(dB\right)=d\left(\partial_t B\right)=0 \Rightarrow \exists\, E\in\Lambda^1 / \partial_t B=-dE$$

Axiomatics

- **Four Maxwell's equations ?**

$$\vec{\nabla}.\vec{B}=0 \Rightarrow \partial_t\left(\vec{\nabla}.\vec{B}\right)=\vec{\nabla}.\left(\partial_t\vec{B}\right)=0 \Rightarrow \partial_t\vec{B} \text{ is a “pure curl” } \partial_t\vec{B}=\vec{\nabla}\times\left(-\vec{E}\right)$$

$$dB=0 \Rightarrow \partial_t\left(dB\right)=d\left(\partial_t B\right)=0 \Rightarrow \exists\, E\in\Lambda^1 / \partial_t B=-dE$$

- **Continuity equation : axiom or spin-off ?**

Axiomatics

JA Heras. Am J Phys, **75** 652 (2007)

- **Four Maxwell's equations ?**

$$\vec{\nabla}.\vec{B}=0 \Rightarrow \partial_t\left(\vec{\nabla}.\vec{B}\right)=\vec{\nabla}.\left(\partial_t\vec{B}\right)=0 \Rightarrow \ \partial_t\vec{B} \text{ is a “pure curl” } \partial_t\vec{B}=\vec{\nabla}\times\left(-\vec{E}\right)$$

$$dB=0 \Rightarrow \partial_t\left(dB\right)=d\left(\partial_t B\right)=0 \ \Rightarrow \exists\ E\in\Lambda^1 / \partial_t B=-dE$$

- **Continuity equation : axiom or spin-off ?**

Can Maxwell's equations be obtained from the continuity equation?

José A. Heras[a)]
Departamento de Física y Matemáticas, Universidad Iberoamericana, Prolongación Paseo de la Reforma 880, Mexico D. F. 01210, Mexico and Department of Physics and Astronomy, Louisiana State University, Baton Rouge, Louisiana 70803-4001

(Received 27 March 2006; accepted 20 April 2007)

We formulate an existence theorem that states that, given localized scalar and vector time-dependent sources satisfying the continuity equation, there exist two retarded fields that satisfy a set of four field equations. If the theorem is applied to the usual electromagnetic charge and current densities, the retarded fields are identified with the electric and magnetic fields and the associated field equations with Maxwell's equations. This application of the theorem suggests that charge conservation can be considered to be the fundamental assumption underlying Maxwell's equations.
[DOI: 10.1119/1.2739570]

Axiomatics

JA Heras. Am J Phys, **75** 652 (2007)

- **Four Maxwell's equations ?**

$$\vec{\nabla}.\vec{B}=0 \Rightarrow \partial_t\left(\vec{\nabla}.\vec{B}\right)=\vec{\nabla}.\left(\partial_t\vec{B}\right)=0 \Rightarrow \ \partial_t\vec{B} \text{ is a "pure curl" } \partial_t\vec{B}=\vec{\nabla}\times\left(-\vec{E}\right)$$

$$dB=0 \Rightarrow \partial_t\left(dB\right)=d\left(\partial_t B\right)=0 \ \Rightarrow \exists\, E\in\Lambda^1 / \partial_t B=-dE$$

- **Continuity equation : axiom or spin-off ?**

In this section we formulate and demonstrate the following existence theorem: Given the localized sources $\rho(\mathbf{x},t)$ and $\mathbf{J}(\mathbf{x},t)$ which satisfy the continuity equation,

$$\nabla\cdot\mathbf{J}+\frac{\partial\rho}{\partial t}=0, \tag{1}$$

there exist retarded fields $\mathbf{F}(\mathbf{x},t)$ and $\mathbf{G}(\mathbf{x},t)$ defined by

$$\mathbf{F}=\frac{\alpha}{4\pi}\int d^3x'\left(\frac{\hat{\mathbf{R}}}{R^2}[\rho]+\frac{\hat{\mathbf{R}}}{Rc}\left[\frac{\partial\rho}{\partial t}\right]-\frac{1}{Rc^2}\left[\frac{\partial\mathbf{J}}{\partial t}\right]\right), \tag{2a}$$

$$\mathbf{G}=\frac{\beta}{4\pi}\int d^3x'\left([\mathbf{J}]\times\frac{\hat{\mathbf{R}}}{R^2}+\left[\frac{\partial\mathbf{J}}{\partial t}\right]\times\frac{\hat{\mathbf{R}}}{Rc}\right), \tag{2b}$$

that satisfy the field equations

$$\nabla\cdot\mathbf{F}=\alpha\rho, \tag{3a}$$

$$\nabla\cdot\mathbf{G}=0, \tag{3b}$$

$$\nabla\times\mathbf{F}+\gamma\frac{\partial\mathbf{G}}{\partial t}=0, \tag{3c}$$

$$\nabla\times\mathbf{G}-\frac{\beta}{\alpha}\frac{\partial\mathbf{F}}{\partial t}=\beta\mathbf{J}. \tag{3d}$$

Axiomatics

JA Heras. Am J Phys, **75** 652 (2007)

- **Four Maxwell's equations ?**

$$\vec{\nabla}.\vec{B} = 0 \Rightarrow \partial_t\left(\vec{\nabla}.\vec{B}\right) = \vec{\nabla}.\left(\partial_t\vec{B}\right) = 0 \Rightarrow \partial_t\vec{B} \text{ is a “pure curl” } \partial_t\vec{B} = \vec{\nabla}\times\left(-\vec{E}\right)$$

$$dB = 0 \Rightarrow \partial_t\left(dB\right) = d\left(\partial_t B\right) = 0 \Rightarrow \exists\, E \in \Lambda^1 / \partial_t B = -dE$$

- **Continuity equation : axiom or spin-off ?**

- **«Contamination» by gravity**

Axiomatics

JA Heras. Am J Phys, **75** 652 (2007)

- **Four Maxwell's equations ?**

$$\vec{\nabla}.\vec{B}=0 \Rightarrow \partial_t\left(\vec{\nabla}.\vec{B}\right)=\vec{\nabla}.\left(\partial_t\vec{B}\right)=0 \Rightarrow \quad \partial_t\vec{B} \text{ is a "pure curl" } \partial_t\vec{B}=\vec{\nabla}\times\left(-\vec{E}\right)$$

$$dB=0 \Rightarrow \partial_t\left(dB\right)=d\left(\partial_t B\right)=0 \quad \Rightarrow \exists\, E\in\Lambda^1 / \partial_t B=-dE$$

- **Continuity equation : axiom or spin-off ?**

- **«Contamination» by gravity**

  - Dual constitutive relations (Hodge implies metric)

  - Retardation time assumes a metric structure (distance, causality)

Axiomatics

JA Heras. Am J Phys, **75** 652 (2007)

- **Four Maxwell's equations ?**

$$\vec{\nabla}.\vec{B}=0 \Rightarrow \partial_t\left(\vec{\nabla}.\vec{B}\right)=\vec{\nabla}.\left(\partial_t\vec{B}\right)=0 \Rightarrow \quad \partial_t\vec{B} \text{ is a “pure curl” } \partial_t\vec{B}=\vec{\nabla}\times\left(-\vec{E}\right)$$

$$dB=0 \Rightarrow \partial_t\left(dB\right)=d\left(\partial_t B\right)=0 \quad \Rightarrow \exists\, E\in\Lambda^1 / \partial_t B=-dE$$

- **Continuity equation : axiom or spin-off ?**

- **«Contamination» by gravity**

  - Dual constitutive relations (Hodge implies metric)
  - Retardation time assumes a metric structure (distance, causality)

"Indeed, the notion of metric is a very complicated one: it requires measurements with clocks and scales, generally with rigid bodies, which are themselves things of extreme complexity. Hence it seems undesirable to take the notion of a metric as a fundament, also of phenomena which are much simpler and independent of it. I might state as a principle, or rather as a program: to formulate the fundamental laws of physics in a form **independent of metrical geometry**."

Van Dantzig, cited in E. Whittaker (1953)

Axiomatics

JA Heras. Am J Phys, **75** 652 (2007)

- **Four Maxwell's equations ?**

$$\vec{\nabla}.\vec{B}=0 \Rightarrow \partial_t\left(\vec{\nabla}.\vec{B}\right)=\vec{\nabla}.\left(\partial_t\vec{B}\right)=0 \Rightarrow \partial_t\vec{B} \text{ is a "pure curl" } \partial_t\vec{B}=\vec{\nabla}\times\left(-\vec{E}\right)$$

$$dB=0 \Rightarrow \partial_t\left(dB\right)=d\left(\partial_t B\right)=0 \Rightarrow \exists\, E\in\Lambda^1 / \partial_t B=-dE$$

- **Continuity equation : axiom or spin-off ?**

- **«Contamination» by gravity**

  - Dual constitutive relations (Hodge implies metric)
  - Retardation time assumes a metric structure (distance, causality)

↪ **From what minimal set of non-metric equations a purely EM theory can be built ?**

1. 3+1

The hydrogen atom

P. Ehrenfest. Ann. Physik, **61** 440 (1920)
I. Freeman. Am. J. Phys., **37** 1222 (1969)

- ***n*-dimensional Poisson equation for a point charge** : $$\sum_{i=1}^{n}\frac{\partial^2\Phi}{\partial x_i^2}=-q\delta\left(r\right)\qquad\left(n\in\mathbb{N}\right)$$

The hydrogen atom

P. Ehrenfest. Ann. Physik, **61** 440 (1920)
I. Freeman. Am. J. Phys., **37** 1222 (1969)

- ***n*-dimensional Poisson equation for a point charge** : $\sum_{i=1}^{n} \frac{\partial^2 \Phi}{\partial x_i^2} = -q\delta(r) \qquad (n \in \mathbb{N})$

Assume $\Phi(r)$ where $r = \sqrt{\sum_{i=1}^{n} x_i^2}$, then Poisson equation simplifies into:

$$\frac{d^2\Phi}{dr^2} + \frac{n-1}{r}\Phi = -q\delta(r)$$

The hydrogen atom

P. Ehrenfest. Ann. Physik, **61** 440 (1920)
I. Freeman. Am. J. Phys., **37** 1222 (1969)

- ***n*-dimensional Poisson equation for a point charge** : $\sum_{i=1}^{n} \frac{\partial^2 \Phi}{\partial x_i^2} = -q\delta(r) \qquad (n \in \mathbb{N})$

Assume $\Phi(r)$ where $r = \sqrt{\sum_{i=1}^{n} x_i^2}$, then Poisson equation simplifies into:

$$\frac{d^2\Phi}{dr^2} + \frac{n-1}{r}\Phi = -q\delta(r) \quad \Rightarrow V(r) = q\,\Phi(r) = -\frac{e^2}{r^{n-2}}$$

The hydrogen atom

P. Ehrenfest. Ann. Physik, **61** 440 (1920)
I. Freeman. Am. J. Phys., **37** 1222 (1969)

- ***n*-dimensional Poisson equation for a point charge** : $\sum_{i=1}^{n} \frac{\partial^2 \Phi}{\partial x_i^2} = -q\delta(r) \qquad (n \in \mathbb{N})$

Assume $\Phi(r)$ where $r = \sqrt{\sum_{i=1}^{n} x_i^2}$, then Poisson equation simplifies into:

$$\frac{d^2\Phi}{dr^2} + \frac{n-1}{r}\Phi = -q\delta(r) \quad \Rightarrow V(r) = q\,\Phi(r) = -\frac{e^2}{r^{n-2}}$$

- **Hydrogen atom** : $E = \frac{p^2}{2m} + V(r) \approx \frac{\hbar^2}{2mr^2} - \frac{e^2}{r^{n-2}}$ (Heisenberg uncertainty principle)

The hydrogen atom

P. Ehrenfest. Ann. Physik, **61** 440 (1920)
I. Freeman. Am. J. Phys., **37** 1222 (1969)

- ***n*-dimensional Poisson equation for a point charge** : $\sum_{i=1}^{n} \frac{\partial^2 \Phi}{\partial x_i^2} = -q\delta(r)$ $\quad (n \in \mathbb{N})$

Assume $\Phi(r)$ where $r = \sqrt{\sum_{i=1}^{n} x_i^2}$, then Poisson equation simplifies into:

$$\frac{d^2\Phi}{dr^2} + \frac{n-1}{r}\Phi = -q\delta(r) \quad \Rightarrow V(r) = q\,\Phi(r) = -\frac{e^2}{r^{n-2}}$$

- **Hydrogen atom** : $E = \frac{p^2}{2m} + V(r) \approx \frac{\hbar^2}{2mr^2} - \frac{e^2}{r^{n-2}}$ (Heisenberg uncertainty principle)

$$\frac{dE_{\min}}{dr} = 0 \Rightarrow r_0 = \left(\frac{\hbar^2}{(n-2)me^2}\right)^{\frac{1}{4-n}}$$

The hydrogen atom

P. Ehrenfest. Ann. Physik, **61** 440 (1920)
I. Freeman. Am. J. Phys., **37** 1222 (1969)

- ***n*-dimensional Poisson equation for a point charge** : $\sum_{i=1}^{n} \frac{\partial^2 \Phi}{\partial x_i^2} = -q\delta(r) \qquad (n \in \mathbb{N})$

Assume $\Phi(r)$ where $r = \sqrt{\sum_{i=1}^{n} x_i^2}$, then Poisson equation simplifies into:

$$\frac{d^2\Phi}{dr^2} + \frac{n-1}{r}\Phi = -q\delta(r) \quad \Rightarrow V(r) = q\,\Phi(r) = -\frac{e^2}{r^{n-2}}$$

- **Hydrogen atom** : $E = \frac{p^2}{2m} + V(r) \approx \frac{\hbar^2}{2mr^2} - \frac{e^2}{r^{n-2}}$ (Heisenberg uncertainty principle)

$$\frac{dE_{\min}}{dr} = 0 \Rightarrow r_0 = \left(\frac{\hbar^2}{(n-2)me^2}\right)^{\frac{1}{4-n}}$$

Orbits only defined when $\boldsymbol{n > 2}$

The hydrogen atom

P. Ehrenfest. Ann. Physik, **61** 440 (1920)
I. Freeman. Am. J. Phys., **37** 1222 (1969)

- ***n*-dimensional Poisson equation for a point charge** : $\sum_{i=1}^{n} \frac{\partial^2 \Phi}{\partial x_i^2} = -q\delta(r) \qquad (n \in \mathbb{N})$

Assume $\Phi(r)$ where $r = \sqrt{\sum_{i=1}^{n} x_i^2}$, then Poisson equation simplifies into:

$$\frac{d^2\Phi}{dr^2} + \frac{n-1}{r}\Phi = -q\delta(r) \quad \Rightarrow V(r) = q\,\Phi(r) = -\frac{e^2}{r^{n-2}}$$

- **Hydrogen atom** : $E = \frac{p^2}{2m} + V(r) \approx \frac{\hbar^2}{2mr^2} - \frac{e^2}{r^{n-2}}$ (Heisenberg uncertainty principle)

$$\frac{dE_{\min}}{dr} = 0 \Rightarrow r_0 = \left(\frac{\hbar^2}{(n-2)me^2}\right)^{\frac{1}{4-n}} \qquad \left.\frac{d^2E_{\min}}{dr^2}\right)_{r=r_0} = \frac{\hbar^2}{mr_0^4}(4-n) > 0$$

Orbits only defined when $n > 2$

The hydrogen atom

P. Ehrenfest. Ann. Physik, **61** 440 (1920)
I. Freeman. Am. J. Phys., **37** 1222 (1969)

- ***n*-dimensional Poisson equation for a point charge** : $\sum_{i=1}^{n} \frac{\partial^2 \Phi}{\partial x_i^2} = -q\delta(r) \qquad (n \in \mathbb{N})$

  Assume $\Phi(r)$ where $r = \sqrt{\sum_{i=1}^{n} x_i^2}$, then Poisson equation simplifies into:

$$\frac{d^2\Phi}{dr^2} + \frac{n-1}{r}\Phi = -q\delta(r) \quad \Rightarrow V(r) = q\,\Phi(r) = -\frac{e^2}{r^{n-2}}$$

- **Hydrogen atom** : $E = \frac{p^2}{2m} + V(r) \approx \frac{\hbar^2}{2mr^2} - \frac{e^2}{r^{n-2}}$ (Heisenberg uncertainty principle)

$$\frac{dE_{\min}}{dr} = 0 \Rightarrow r_0 = \left(\frac{\hbar^2}{(n-2)me^2}\right)^{\frac{1}{4-n}}$$

$$\left.\frac{d^2E_{\min}}{dr^2}\right)_{r=r_0} = \frac{\hbar^2}{mr_0^4}(4-n) > 0$$

Orbits only defined when $n > 2$

Energy minimum only when $n < 4$

The hydrogen atom

P. Ehrenfest. Ann. Physik, **61** 440 (1920)
I. Freeman. Am. J. Phys., **37** 1222 (1969)

- ***n*-dimensional Poisson equation for a point charge** : $\sum_{i=1}^{n} \frac{\partial^2 \Phi}{\partial x_i^2} = -q\delta(r) \qquad (n \in \mathbb{N})$

Assume $\Phi(r)$ where $r = \sqrt{\sum_{i=1}^{n} x_i^2}$, then Poisson equation simplifies into:

$$\frac{d^2\Phi}{dr^2} + \frac{n-1}{r}\Phi = -q\delta(r) \quad \Rightarrow \; V(r) = q\,\Phi(r) = -\frac{e^2}{r^{n-2}}$$

- **Hydrogen atom** : $E = \frac{p^2}{2m} + V(r) \approx \frac{\hbar^2}{2mr^2} - \frac{e^2}{r^{n-2}}$ (Heisenberg uncertainty principle)

$$\frac{dE_{\min}}{dr} = 0 \Rightarrow r_0 = \left(\frac{\hbar^2}{(n-2)me^2}\right)^{\frac{1}{4-n}}$$

$$\left.\frac{d^2E_{\min}}{dr^2}\right)_{r=r_0} = \frac{\hbar^2}{mr_0^4}(4-n) > 0$$

Orbits only defined when $n > 2$

Energy minimum only when $n < 4$

stable state for hydrogen atom $\Rightarrow n = 3$

Others tricks

FR Tangherlini. Nuovo Cim. **27** 636 (1963)
CW Misner, KS Thorne, JA Wheeler. Gravitation (1973)

- **Analog argument to account for stability of planetary orbits**
  - classical : Ehrenfest (1920), Büchel (1963).
  - general relativity : Tangherlini (1963), Misner-Thorne-Wheeler p.1205 (1973).

Others tricks

FR Tangherlini. Nuovo Cim. **27** 636 (1963)
CW Misner, KS Thorne, JA Wheeler. Gravitation (1973)

- **Analog argument to account for stability of planetary orbits**
  - classical : Ehrenfest (1920), Büchel (1963).
  - general relativity : Tangherlini (1963), Misner-Thorne-Wheeler p.1205 (1973).

- **Thermodynamics**

epl A LETTERS JOURNAL EXPLORING THE FRONTIERS OF PHYSICS
February 2016
EPL, **113** (2016) 40006
doi: 10.1209/0295-5075/113/40006
www.epljournal.org

Is the $(3+1)$-d nature of the universe a thermodynamic necessity?

JULIAN GONZALEZ-AYALA[1,2], RUBÉN CORDERO[1] and F. ANGULO-BROWN[1]

[1] *Departamento de Física, Escuela Superior de Física y Matemáticas, Instituto Politécnico Nacional Edif. No. 9, U.P. Zacatenco, 07738, México D. F., México*
[2] *Departamento de Física Aplicada, Universidad de Salamanca - 37001, Salamanca, España*

Others tricks

- **Topology**

Eur. Phys. J. C (2017) 77:682
DOI 10.1140/epjc/s10052-017-5253-3

THE EUROPEAN PHYSICAL JOURNAL C

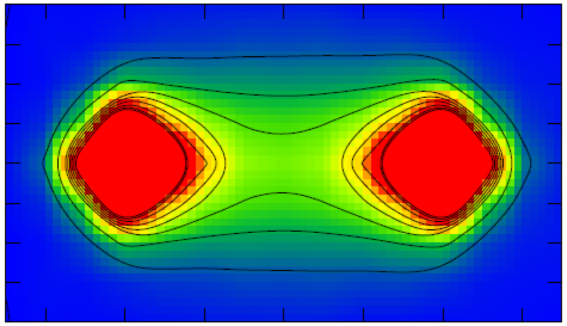
CrossMark

Letter

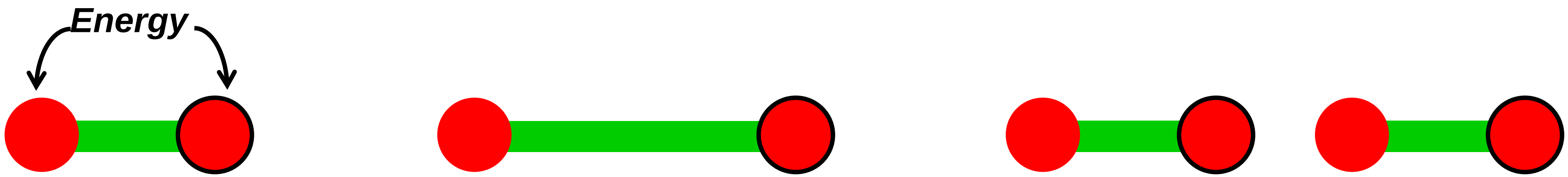
Knotty inflation and the dimensionality of spacetime

Arjun Berera[1,a], Roman V. Buniy[2,b], Thomas W. Kephart[3,c], Heinrich Päs[4,d], João G. Rosa[5,e]

[1] Tait Institute, School of Physics and Astronomy, University of Edinburgh, Edinburgh EH9 3FD, UK
[2] Schmid College of Science, Chapman University, Orange, CA 92866, USA
[3] Department of Physics and Astronomy, Vanderbilt University, Nashville, TN 37235, USA
[4] Fakultät für Physik, Technische Universität Dortmund, 44221 Dortmund, Germany
[5] Departamento de Física da Universidade de Aveiro and CIDMA, Campus de Santiago, 3810-183 Aveiro, Portugal



M. Cardoso et al. PRD 81, (2010)



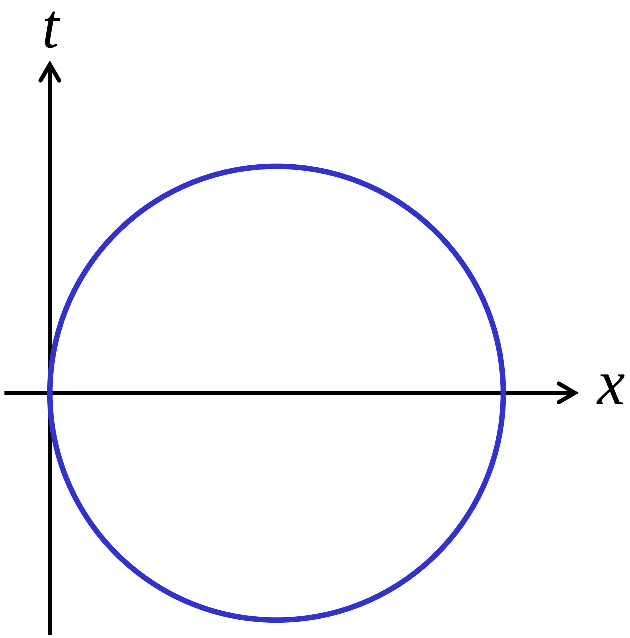
⇔ cutting a magnet

Time-like loops

JG Foster and B Müller, hep-th/1001.2485 (2010)

- **One time dimension:** $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$

$$\textit{Loop} \quad \begin{cases} t = R\sin\alpha \\ x = R(1-\cos\alpha) \end{cases}$$

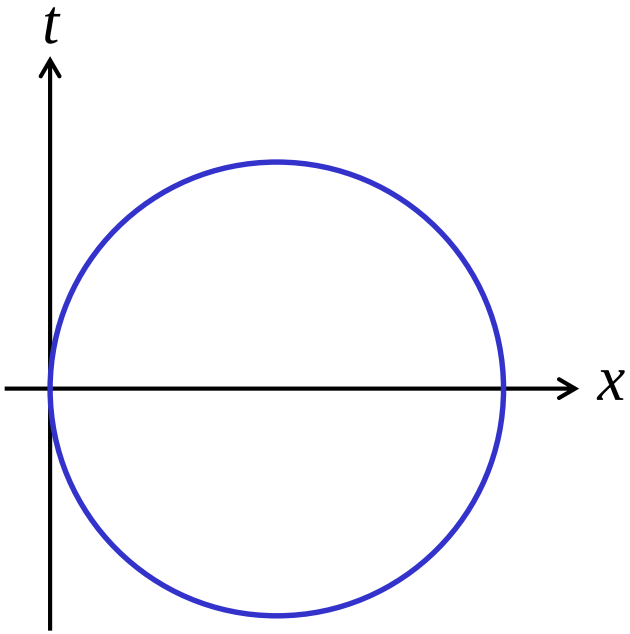


Time-like loops

JG Foster and B Müller, hep-th/1001.2485 (2010)

- **One time dimension:** $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$

Loop $\begin{cases} t = R\sin\alpha \\ x = R(1-\cos\alpha) \end{cases} \Rightarrow ds^2 = -\cos(2\alpha) R^2 d\alpha^2$

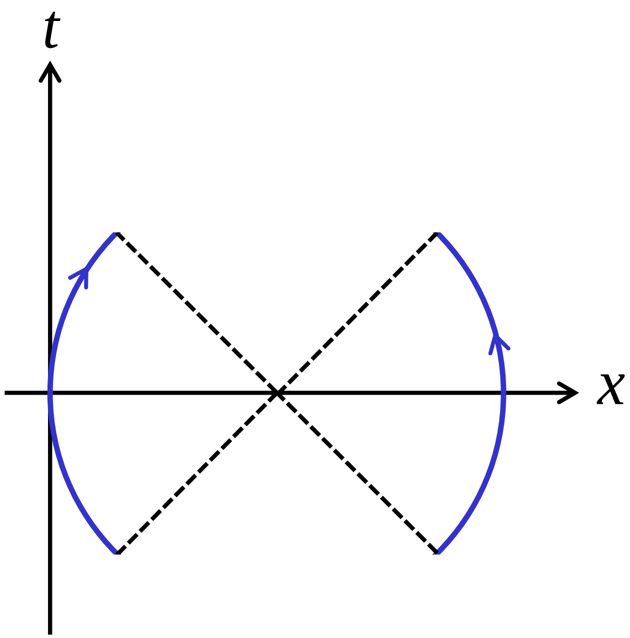


Time-like loops

JG Foster and B Müller, hep-th/1001.2485 (2010)

- **One time dimension:** $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$

Loop $\begin{cases} t = R\sin\alpha \\ x = R(1-\cos\alpha) \end{cases} \Rightarrow ds^2 = -\cos(2\alpha)R^2 d\alpha^2 > 0$ for $\dfrac{\pi}{4} < \alpha < \dfrac{3\pi}{4}$



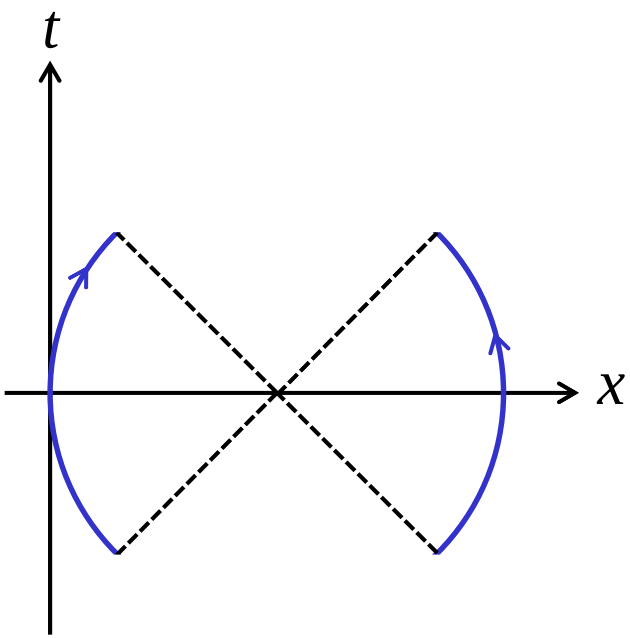
not time-like everywhere !

Time-like loops

JG Foster and B Müller, hep-th/1001.2485 (2010)

- **One time dimension:** $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$

Loop $\begin{cases} t = R\sin\alpha \\ x = R(1-\cos\alpha) \end{cases} \Rightarrow ds^2 = -\cos(2\alpha)R^2 d\alpha^2 > 0$ for $\dfrac{\pi}{4} < \alpha < \dfrac{3\pi}{4}$



not time-like everywhere !

no closed time-loop

Time-like loops

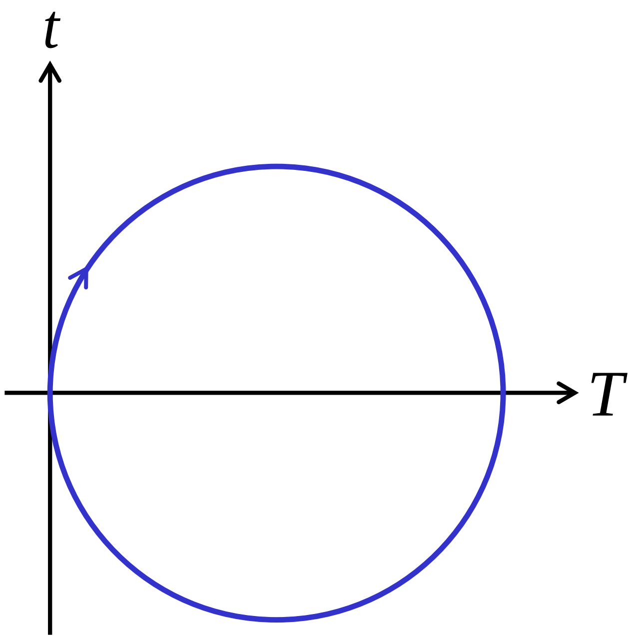
JG Foster and B Müller, hep-th/1001.2485 (2010)

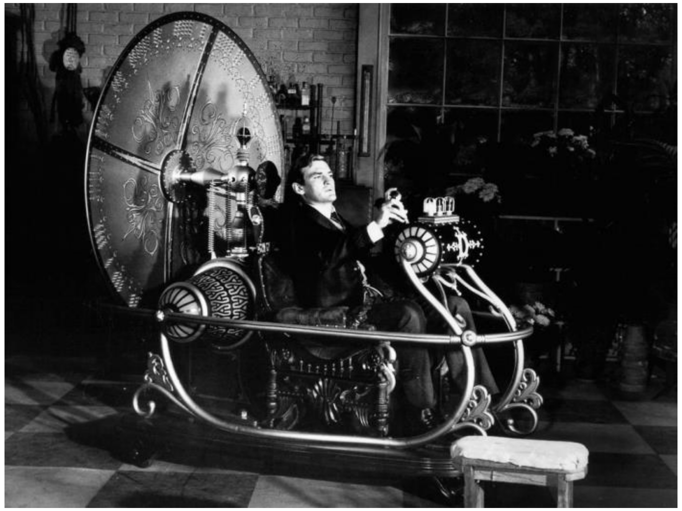
- **Two time dimension:** $ds^2 = -dt^2 - dT^2 + dy^2 + dz^2$

Loop $\begin{cases} t = R\sin\alpha \\ T = R(1-\cos\alpha) \end{cases} \Rightarrow ds^2 = -R^2 d\alpha^2 < 0$

time-like everywhere ➡ **closed time-like loop**

Causality breaking (↔ Lenz's law)

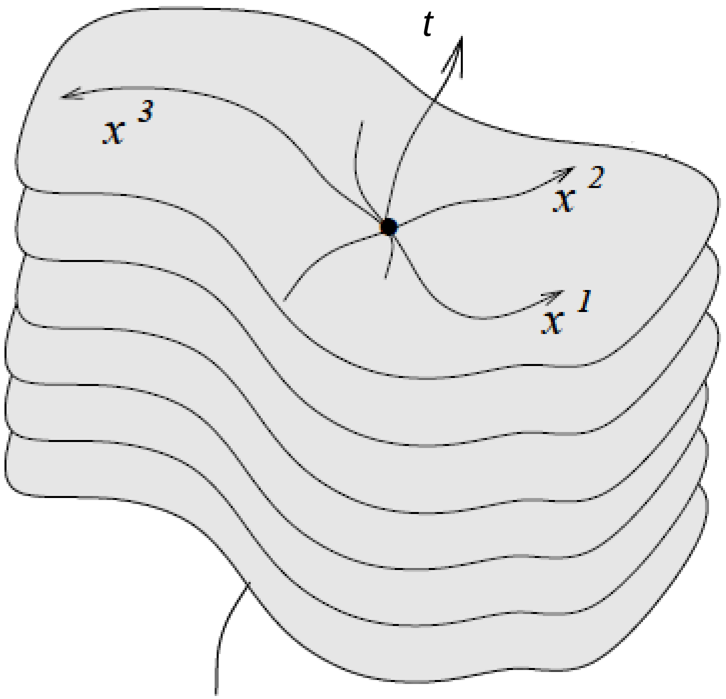




HG Wells, The time machine.

Topological spacetime

- **Spacetime = 4D continuum**
  - Connected, Haussdorff, orientable…
  - Local foliation 3+1: spacetime is sliced in 3D folios labeled by a monotonically increasing a prototime parameter *t* .
  - Bare manifold = **no metric, no connection**



2. Premetric electrodynamics

De Rham cohomology

- A p-form a is closed if $da = 0$. The set of closed p-forms is denoted as $\mathrm{Z}^p(\mathcal{M})$. A p-form *a* is exact if there is a p-1 form b such that $a = db$. The set of exact p-forms is denoted as $\mathrm{B}^p(\mathcal{M})$.

$$\mathrm{B}^p(\mathcal{M}) \subset \mathrm{Z}^p(\mathcal{M})$$

- Conversely, when is a closed p-form exact ? De Rham theorem.
  - *Locally, always (Poincaré's lemma).*
  - *Globally, it depends on the topology of $\mathcal{M}$. Defining De Rham p^th cohomology group:*

$$\mathrm{H}^p(\mathcal{M}) = \mathrm{Z}^p(\mathcal{M}) \,/\, \mathrm{B}^p(\mathcal{M})$$

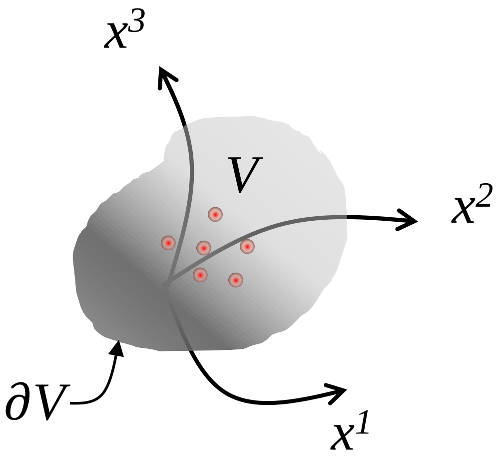
dim $(\mathrm{H}^p(\mathcal{M})) = \mathrm{b}_p$ = pth Betti number $\leftrightarrow \chi$
= number of « p-holes »

↳ if $\mathrm{b}_p = 0$, then it is also exact.

↳ if $\mathrm{b}_p \neq 0$, then a closed p-form is exact if and only if all of its periods vanish.

Locality of charge

- Counting the number of elementary charges localized inside a 3-dimensional compact domain V (bounded by surface ∂V), an electric charge density ρ can be defined from the total charge Q according to :



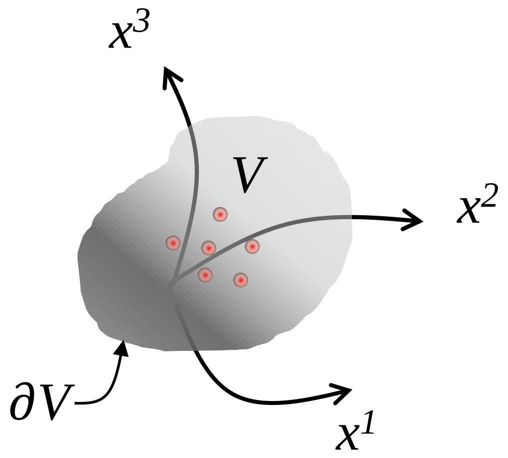
$$Q = \int_V \rho$$

$[\rho]=[Q]=$q

axiom 1

Locality of charge

- Counting the number of elementary charges localized inside a 3-dimensional compact domain V (bounded by surface ∂V), an electric charge density ρ can be defined from the total charge Q according to :



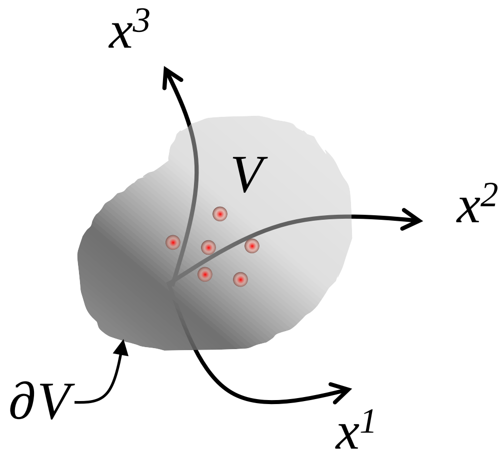
$$Q = \int_V \rho \qquad [\rho]=[Q]=\mathrm{q}$$

axiom 1

$\Rightarrow$ charge density is a 3-form: $\rho = \rho_{123} dx^1 \wedge dx^2 \wedge dx^3$

Locality of charge

• Counting the number of elementary charges localized inside a 3-dimensional compact domain V (bounded by surface ∂V), an electric charge density ρ can be defined from the total charge Q according to :



$$Q = \int_V \rho$$

$[\rho]=[Q]=\mathrm{q}$

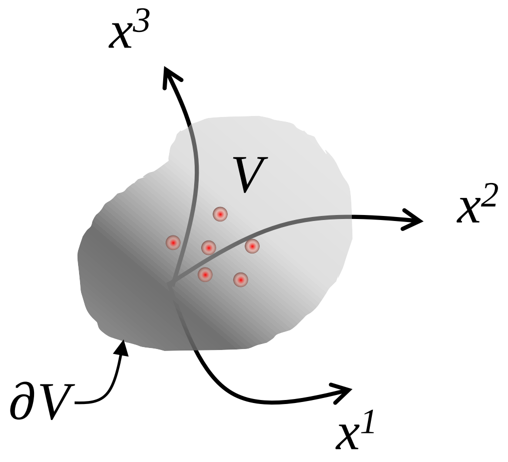
axiom 1

⇒ charge density is a 3-form: $\rho = \rho_{123} dx^1 \wedge dx^2 \wedge dx^3$

⇒ in 3 dimensions, $\boxed{d\rho = 0}$

Locality of charge

• Counting the number of elementary charges localized inside a 3-dimensional compact domain V (bounded by surface ∂V), an electric charge density ρ can be defined from the total charge Q according to :



$$Q = \int_V \rho$$

$[\rho]=[Q]=$q

axiom 1

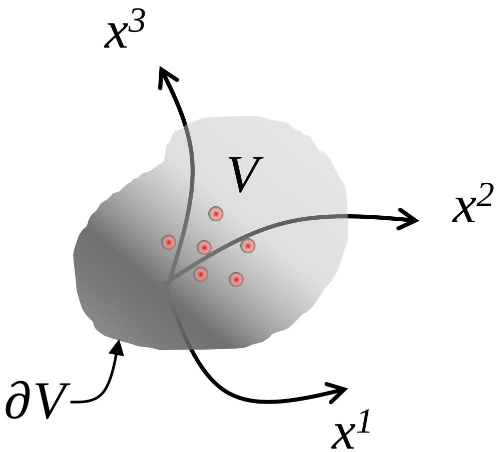
$\Rightarrow$ charge density is a 3-form: $\rho = \rho_{123} dx^1 \wedge dx^2 \wedge dx^3$

$\Rightarrow$ in 3 dimensions, $\boxed{d\rho = 0}$

1) This equation relies only on exterior calculus.

Locality of charge

• Counting the number of elementary charges localized inside a 3-dimensional compact domain V (bounded by surface ∂V), an electric charge density ρ can be defined from the total charge Q according to :



$$Q = \int_V \rho \qquad [\rho]=[Q]=q$$

axiom 1

⇒ charge density is a 3-form: $\rho = \rho_{123} dx^1 \wedge dx^2 \wedge dx^3$

⇒ in 3 dimensions, $\boxed{d\rho = 0}$

1) This equation relies only on exterior calculus.

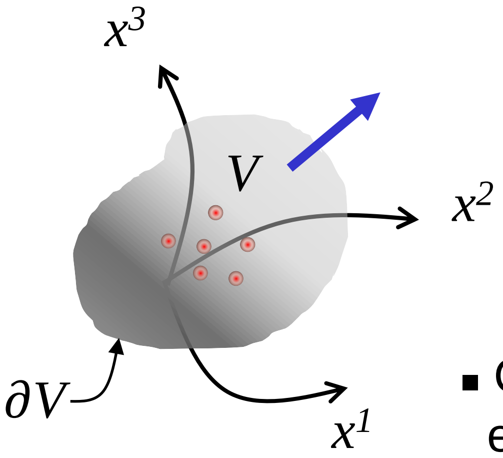
2) No analog relation in Maxwell-Heaviside formulation

Current density

FW Hehl, YN Obukhov. Foundations of classical electrodynamics. (2003)

- Similarly, in 3+1, $\dot{\rho}$ is a 3-form $\Rightarrow$ $d\dot{\rho} = 0$

$\Rightarrow$ De Rham theorem: $\dot{\rho}$ is exact and derives from a 2-form « potential »



$$\boxed{\dot{\rho} = d\left(-j\right)}$$

[j]=q/t

Local charge conservation

- Charges are generally not at rest and j can be interpreted as the electric current density obtained by counting the number of charges escaping through ∂V per unit time.

- The outer orientation and hence the sign of j are chosen such that the charge density decreases when j is outgoing.

Outcomes of charge conservation

Charge density ρ is a 3-form:

⇒ De Rham theorem: if $d\rho = 0$, ∃ a 2-form D such that $\rho = dD$

Premetric
Maxwell-Gauss equation

$$\boxed{dD = \rho}$$

D = electric flux intensity 2-form

[D]=[ρ]=q

$$\dot{\rho} + dj = 0 \;\Rightarrow\; d\left(\dot{D} + j\right) = 0$$

⇒ De Rham theorem: ∃ a 1-form H such that

Premetric
Maxwell-Ampère equation

$$\boxed{\dot{D} + j = dH}$$

H = magnetic field intensity 1-form

[H]=[j]=q/t

Maxwell's inhomogeneous equations

Status of excitations

FW Hehl, YN Obukhov. Foundations of classical electrodynamics. (2003)

Electric charge conservation is valid in microphysics. Therefore the corresponding Maxwell equations (3.3) and (3.4) are valid on the same "microphysical" level as the notions of charge density ρ and current density j. And with them the excitations $\mathcal{D}$ and $\mathcal{H}$ are microphysical quantities of the same type likewise – in contrast to what is stated in most textbooks.

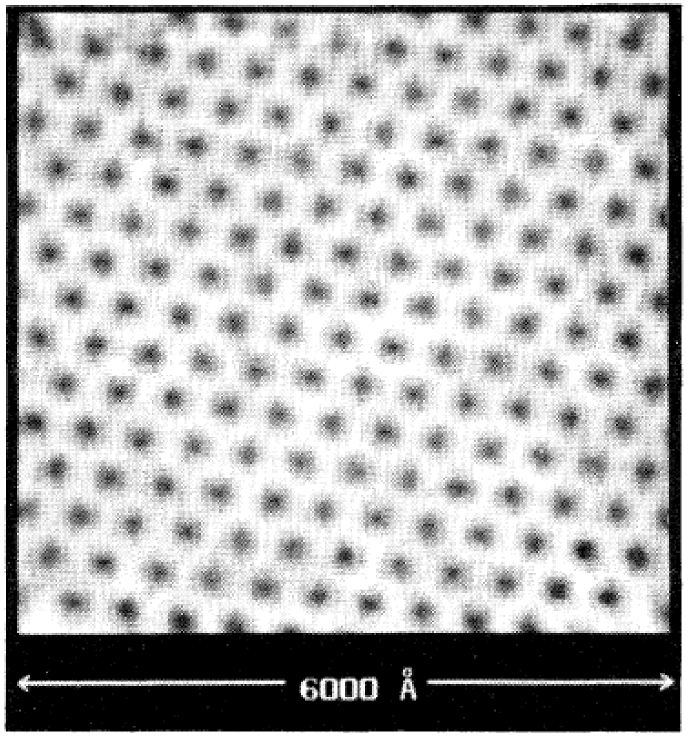
FW Hehl, YN Obukhov. ArXiv:physics/0005084

⇒ **In premetric electrodynamics**,

- D and H are **fundamental potentials** associated to the sources (ρ, j) and they are relevant not only in matter but also in vacuum.
- They do not come from Lorentz-Rosenfeld averaging processes (polarization, magnetization) but from a charge counting procedure, i.e. they are **microscopic fields**.

Flux conservation

HF Hess et al. PRL, **62** 214 (1989)
FW Hehl, YN Obukhov. Foundations of classical electrodynamics. (2003)



Abrikosov flux lines lattice in $NbSe_2$ (1T, 1.8 K)

- Quantized magnetic flux lines can be counted and behave like a conserved quantity = they move without being created or destroyed

⇒ Similarly to charge conservation, one postulates the existence of a density of flux lines, the 2-form magnetic flux density B :

$$\Phi = \int_S B$$

axiom 2

$$\boxed{dB = 0}$$

Premetric Maxwell-Thomson equation

Outcomes of flux conservation

$\Rightarrow$ De Rham theorem: $\exists$ a 1-form E such that

$$\boxed{\dot{B} + dE = 0}$$

$$\updownarrow \qquad \updownarrow$$

$$\dot{\rho} + dj = 0$$

Premetric Maxwell-Faraday equation

- As for charge conservation, the outer orientation and hence the sign of E are chosen such that the magnetic flux decreases when B is outgoing.

- The magnetic flux density (also known as magnetic field strength) B carries the dimension [B]=h/q and hence, the <u>current density of magnetic flux </u>(also known as electric field strength) E has dimension [E]=h/(tq).

Remarks

1. Only two axioms are needed to recover Maxwell's equations. Charge conservation is an outcome of axiom 1.

2. Charge conservation requires a counting procedure of elementary charges, a way to delimit an arbitrary 3-volume V by a boundary ∂V and a way to know what is inside and what is outside ∂V. Hence, it does not rely the concept of length and it is purely **topological**.

3. The counting procedure makes charge and flux conservations **valid at a microscopic level**.

4. Anytime a physical quantity is represented by an closed form, there must be a local conservation law associated to it.

5. Axioms of premetric EM rule out the possibility of massive photons…

3. Emergence of spacetime

Closure of EM field equations

- In 4D, the axioms on EM fields write as $\left(\varepsilon_0 = \mu_0 = c = 1\right)$

$$d_4 G = J \qquad d_4 F = 0$$

with

$$G = D - H \wedge dt$$
$$F = E \wedge dt + B$$
$$J = \rho - j \wedge dt$$

Closure of EM field equations

- In 4D, the axioms on EM fields write as $d_4 G = J$ with $G = D - H \wedge dt$

  $(\varepsilon_0 = \mu_0 = c = 1)$ $d_4 F = 0$ $F = E \wedge dt + B$

  $J = \rho - j \wedge dt$

- Topological equations relies on 2 fundamental fields F, G but
  - there are 12 independent unknowns
  - there are only 8 independent equations

  ⇒ premetric equations are underdetermined system.

  ⇒ closing the set = relating strengths and excitations

Closure of EM field equations

- In 4D, the axioms on EM fields write as $(\varepsilon_0 = \mu_0 = c = 1)$

$$d_4 G = J \qquad d_4 F = 0$$

with

$$G = D - H \wedge dt \qquad F = E \wedge dt + B \qquad J = \rho - j \wedge dt$$

- Topological equations relies on 2 fundamental fields F, G but
  - there are 12 independent unknowns
  - there are only 8 independent equations

⇒ premetric equations are underdetermined system.

⇒ closing the set = relating strengths and excitations

- Spacetime = bare manifold = neither metric nor connection ⇒ no Hodge operator

Closure of EM field equations

- In 4D, the axioms on EM fields write as $d_4 G = J$ with $G = D - H \wedge dt$

  $(\varepsilon_0 = \mu_0 = c = 1)$ $d_4 F = 0$ $F = E \wedge dt + B$

  $J = \rho - j \wedge dt$

- Topological equations relies on 2 fundamental fields F, G but
  - there are 12 independent unknowns
  - there are only 8 independent equations

  ⇒ premetric equations are underdetermined system.

  ⇒ closing the set = relating strengths and excitations

- Spacetime = bare manifold = neither metric nor connection ⇒ no Hodge operator

The trick : F and G are forms of the same degree ⇒ **linear and local relation**

Axion, dilaton, skewon

G Rubilar. Ann. Phys. **514** 10 (2002)

axiom 3 $G = \# \ F$

Topological constitutive relation in vacuum

$$\boxed{G_{\mu\nu} = \frac{1}{4}\varepsilon_{\mu\nu\alpha\beta}\chi^{\alpha\beta\delta\gamma}F_{\delta\gamma}}$$

constitutive tensor density
36 independant components

$$\chi^{\alpha\beta\delta\gamma} = -\chi^{\beta\alpha\delta\gamma} = -\chi^{\alpha\beta\gamma\delta}$$

Axion, dilaton, skewon

G Rubilar. Ann. Phys. **514** 10 (2002)

axiom 3 $G = \# F$

Topological constitutive relation in vacuum

$$G_{\mu\nu} = \frac{1}{4}\varepsilon_{\mu\nu\alpha\beta}\chi^{\alpha\beta\delta\gamma}F_{\delta\gamma}$$

constitutive tensor density
36 independant components

$$\chi^{\alpha\beta\delta\gamma} = -\chi^{\beta\alpha\delta\gamma} = -\chi^{\alpha\beta\gamma\delta}$$

- Decomposition of the constitutive tensor $\chi^{\alpha\beta\delta\gamma} = {}^{(p)}\chi^{\alpha\beta\delta\gamma} + {}^{(s)}\chi^{\alpha\beta\delta\gamma} + {}^{(a)}\chi^{\alpha\beta\delta\gamma}$

$${}^{(a)}\chi^{\alpha\beta\delta\gamma} = \chi^{[\alpha\beta\delta\gamma]}$$

$${}^{(s)}\chi^{\alpha\beta\delta\gamma} = \frac{1}{2}\left(\chi^{\delta\gamma\alpha\beta} - \chi^{\alpha\beta\delta\gamma}\right)$$

$${}^{(p)}\chi^{\alpha\beta\delta\gamma} = \chi^{\alpha\beta\delta\gamma} - {}^{(a)}\chi^{\alpha\beta\delta\gamma} - {}^{(s)}\chi^{\alpha\beta\delta\gamma}$$

Axion, dilaton, skewon

G Rubilar. Ann. Phys. **514** 10 (2002)
FW Hehl et al. Phys. Rev. A **77** (2008)

axiom 3 $G = \# F$

Topological constitutive relation in vacuum

$$\boxed{G_{\mu\nu} = \frac{1}{4}\varepsilon_{\mu\nu\alpha\beta}\chi^{\alpha\beta\delta\gamma}F_{\delta\gamma}}$$

constitutive tensor density
36 independant components

$$\chi^{\alpha\beta\delta\gamma} = -\chi^{\beta\alpha\delta\gamma} = -\chi^{\alpha\beta\gamma\delta}$$

- Decomposition of the constitutive tensor $\chi^{\alpha\beta\delta\gamma} = {}^{(p)}\chi^{\alpha\beta\delta\gamma} + {}^{(s)}\chi^{\alpha\beta\delta\gamma} + {}^{(a)}\chi^{\alpha\beta\delta\gamma}$

1	${}^{(a)}\chi^{\alpha\beta\delta\gamma} = \chi^{[\alpha\beta\delta\gamma]}$	**Axion (Cr_2O_3*, CDM…)**
15	${}^{(s)}\chi^{\alpha\beta\delta\gamma} = \frac{1}{2}\left(\chi^{\delta\gamma\alpha\beta} - \chi^{\alpha\beta\delta\gamma}\right)$	**Skewon (vacuum polarization)**
20	${}^{(p)}\chi^{\alpha\beta\delta\gamma} = \chi^{\alpha\beta\delta\gamma} - {}^{(a)}\chi^{\alpha\beta\delta\gamma} - {}^{(s)}\chi^{\alpha\beta\delta\gamma}$	**Principal part**

Dispersion relation

Y Obukhov, T Fukui, G Rubilar, Phys. Rev. D 62 044050 (2000)

- Eikonal approximation*:

⇒ **generalized Fresnel equation**: $\boxed{\Omega^{\alpha\beta\delta\gamma} K_\alpha K_\beta K_\delta K_\gamma = 0}$ $\quad K_\mu = (\omega, k_i) \quad i = 1,2,3$

$$\Omega^{\alpha\beta\delta\gamma} = \frac{1}{4!} \varepsilon_{\eta\lambda\nu\theta} \varepsilon_{\tau\omega\sigma\xi} \chi^{\eta\lambda\tau(\alpha} \chi^{\beta|\omega\nu|\delta} \chi^{\gamma)\theta\sigma\xi} = \Omega^{\alpha\beta\delta\gamma}\left({}^{(s)}\chi + {}^{(p)}\chi\right)$$

Tamm-Rubilar tensor density

⇒ quartic surface:

$$\Omega^{\alpha\beta\delta\gamma} K_\alpha K_\beta K_\delta K_\gamma = 0 \Rightarrow M_0\omega^4 + M_1\omega^3 + M_2\omega^2 + M_3\omega + M_4 = 0$$

$$M_0 = \Omega^{0000}$$
$$M_1 = 4\Omega^{000i} k_i$$
$$M_2 = 6\Omega^{00ij} k_i k_j$$
$$M_3 = 4\Omega^{0ijm} k_i k_j k_m$$
$$M_4 = \Omega^{ijmn} k_i k_j k_m k_n$$

Dispersion relation

Y Itin, Phys. Rev. D **72** 087502 (2005)

$$0 = M_0\omega^4 + M_1\omega^3 + M_2\omega^2 + M_3\omega + M_4 = M_0\left(\omega^2 + a\omega + b\right)\left(\omega^2 + c\omega + d\right)$$

Dispersion relation

Y Itin, Phys. Rev. D **72** 087502 (2005)
Polarbear collaboration. Phys. Rev. D **92**, 123509 (2015)

$$0 = M_0\omega^4 + M_1\omega^3 + M_2\omega^2 + M_3\omega + M_4 = M_0\left(\omega^2 + a\omega + b\right)\left(\omega^2 + c\omega + d\right)$$

no vacuum birefringence*= unique light cone = no skewon

Dispersion relation

Y Itin, Phys. Rev. D **72** 087502 (2005)
Polarbear collaboration. Phys. Rev. D **92**, 123509 (2015)

$$0 = M_0\omega^4 + M_1\omega^3 + M_2\omega^2 + M_3\omega + M_4 = M_0\left(\omega^2 + a\omega + b\right)\left(\omega^2 + c\omega + d\right)$$

no vacuum birefringence*= unique light cone = no skewon

- Unique real solution if and only if
  - Uniqueness:

$$M_0\left(\omega^2 + a\omega + b\right)^2 = M_0\omega^4 + M_1\omega^3 + M_2\omega^2 + M_3\omega + M_4 = 0$$

Dispersion relation

Y Itin, Phys. Rev. D **72** 087502 (2005)
Polarbear collaboration. Phys. Rev. D **92**, 123509 (2015)

$$0 = M_0\omega^4 + M_1\omega^3 + M_2\omega^2 + M_3\omega + M_4 = M_0\left(\omega^2 + a\omega + b\right)\left(\omega^2 + c\omega + d\right)$$

no vacuum birefringence*= unique light cone = no skewon

- Unique real solution if and only if
  - Uniqueness:

$$M_0\left(\omega^2 + a\omega + b\right)^2 = M_0\omega^4 + M_1\omega^3 + M_2\omega^2 + M_3\omega + M_4 = 0$$

$$\Rightarrow \quad \begin{array}{ll} \dfrac{M_1}{M_0} = 2a & \dfrac{M_2}{M_0} = a^2 + 2b \\ \dfrac{M_3}{M_0} = 2ab & \dfrac{M_4}{M_0} = b^2 \end{array}$$

Dispersion relation

Y Itin, Phys. Rev. D **72** 087502 (2005)
Polarbear collaboration. Phys. Rev. D **92**, 123509 (2015)

$$0 = M_0\omega^4 + M_1\omega^3 + M_2\omega^2 + M_3\omega + M_4 = M_0\left(\omega^2 + a\omega + b\right)\left(\omega^2 + c\omega + d\right)$$

no vacuum birefringence*= unique light cone = no skewon

- Unique real solution if and only if
  - Uniqueness:

$$M_0\left(\omega^2 + a\omega + b\right)^2 = M_0\omega^4 + M_1\omega^3 + M_2\omega^2 + M_3\omega + M_4 = 0$$

$$\Rightarrow \quad \begin{array}{ll} \dfrac{M_1}{M_0} = 2a & \dfrac{M_2}{M_0} = a^2 + 2b \\ \dfrac{M_3}{M_0} = 2ab & \dfrac{M_4}{M_0} = b^2 \end{array} \quad \Rightarrow \quad \begin{array}{l} a = \dfrac{M_1}{2M_0} \\ b = \dfrac{4M_0M_2 - M_1^2}{8M_0^2} \end{array}$$

Recovering a metric

Y Itin, Phys. Rev. D **72** 087502 (2005)

$$\omega^2 + a\omega + b = 0 \qquad a = \frac{M_1}{2M_0} \in \mathbb{R} \qquad b = \frac{4M_0 M_2 - M_1^2}{8M_0^2} \in \mathbb{R}$$

$$M_0 = \Omega^{0000}$$

$$M_1 = 4\Omega^{000i} k_i$$

$$M_2 = 6\Omega^{00ij} k_i k_j$$

$$\Omega^{00\alpha\beta}\left(\chi\right) = \Omega^{00\alpha\beta}\left({}^{(p)}\chi\right)$$

- Reality : $b \leq 0 \Rightarrow 4M_0 M_2 \leq M_1^2$

Recovering a metric

Y Itin, Phys. Rev. D **72** 087502 (2005)

$$\omega^2 + a\omega + b = 0 \qquad a = \frac{M_1}{2M_0} \in \mathbb{R} \qquad b = \frac{4M_0 M_2 - M_1^2}{8M_0^2} \in \mathbb{R}$$

$$M_0 = \Omega^{0000}$$
$$M_1 = 4\Omega^{000i} k_i$$
$$M_2 = 6\Omega^{00ij} k_i k_j$$
$$\Omega^{00\alpha\beta}(\chi) = \Omega^{00\alpha\beta}\left({}^{(p)}\chi\right)$$

- Reality : $b \le 0 \Rightarrow 4M_0 M_2 \le M_1^2$
- Time reversal : $M_1 = M_3 = 0$

$$\Rightarrow \omega^2 + \frac{1}{2}\frac{M_2}{M_0} = \omega^2 + \frac{3\Omega^{00ij}}{M_0} k_i k_j = \eta^{ab} K_a K_b = 0$$

Recovering a metric

Y Itin, Phys. Rev. D **72** 087502 (2005)

$$\omega^2 + a\omega + b = 0 \qquad a = \frac{M_1}{2M_0} \in \mathbb{R} \qquad b = \frac{4M_0 M_2 - M_1^2}{8M_0^2} \in \mathbb{R}$$

$$M_0 = \Omega^{0000}$$
$$M_1 = 4\Omega^{000i} k_i$$
$$M_2 = 6\Omega^{00ij} k_i k_j$$
$$\Omega^{00\alpha\beta}(\chi) = \Omega^{00\alpha\beta}\left({}^{(p)}\chi\right)$$

- Reality : $b \leq 0 \Rightarrow 4M_0 M_2 \leq M_1^2$
- Time reversal : $M_1 = M_3 = 0$

$$\Rightarrow \omega^2 + \frac{1}{2}\frac{M_2}{M_0} = \omega^2 + \frac{3\Omega^{00ij}}{M_0} k_i k_j = \eta^{ab} K_a K_b = 0$$

$$\boxed{\begin{aligned} \eta^{00} &= 1 \\ \eta^{ij} &= \frac{3\Omega^{00ij}}{M_0} \end{aligned}}$$

Metric tensor components

Recovering a metric

Y Itin, Phys. Rev. D **72** 087502 (2005)

$$\omega^2 + a\omega + b = 0 \qquad a = \frac{M_1}{2M_0} \in \mathbb{R} \qquad b = \frac{4M_0M_2 - M_1^2}{8M_0^2} \in \mathbb{R}$$

$$M_0 = \Omega^{0000}$$
$$M_1 = 4\Omega^{000i}k_i$$
$$M_2 = 6\Omega^{00ij}k_ik_j$$
$$\Omega^{00\alpha\beta}(\chi) = \Omega^{00\alpha\beta}\left({}^{(p)}\chi\right)$$

- Reality : $b \le 0 \Rightarrow 4M_0M_2 \le M_1^2$
- Time reversal : $M_1 = M_3 = 0$

$$\Rightarrow \omega^2 + \frac{1}{2}\frac{M_2}{M_0} = \omega^2 + \frac{3\Omega^{00ij}}{M_0}k_ik_j = \eta^{ab}K_aK_b = 0$$

$$\boxed{\eta^{00} = 1 \qquad \eta^{ij} = \frac{3\Omega^{00ij}}{M_0}}$$

Metric tensor components

1) The metric has a lorentzian signature.

Recovering a metric

Y Itin, Phys. Rev. D **72** 087502 (2005)
FW Hehl, YN Obukhov, GF Rubilar. ArXiv:*gr-qc/9911096*, (1999)

$$\omega^2 + a\omega + b = 0 \qquad a = \frac{M_1}{2M_0} \in \mathbb{R} \qquad b = \frac{4M_0 M_2 - M_1^2}{8M_0^2} \in \mathbb{R}$$

$$M_0 = \Omega^{0000}$$
$$M_1 = 4\Omega^{000i} k_i$$
$$M_2 = 6\Omega^{00ij} k_i k_j$$
$$\Omega^{00\alpha\beta}(\chi) = \Omega^{00\alpha\beta}\left({}^{(p)}\chi\right)$$

- Reality : $b \le 0 \Rightarrow 4M_0 M_2 \le M_1^2$
- Time reversal : $M_1 = M_3 = 0$

$$\Rightarrow \omega^2 + \frac{1}{2}\frac{M_2}{M_0} = \omega^2 + \frac{3\Omega^{00ij}}{M_0} k_i k_j = \eta^{ab} K_a K_b = 0$$

$$\boxed{\eta^{00} = 1 \qquad \eta^{ij} = \frac{3\Omega^{00ij}}{M_0}}$$

Metric tensor components

1) The metric has a lorentzian signature.

2) Compatible with Hodge duality*.

Recovering a metric

Y Itin, Phys. Rev. D **72** 087502 (2005)
FW Hehl, YN Obukhov, GF Rubilar. ArXiv:*gr-qc/9911096*, (1999)

$$\omega^2 + a\omega + b = 0 \qquad a = \frac{M_1}{2M_0} \in \mathbb{R} \qquad b = \frac{4M_0M_2 - M_1^2}{8M_0^2} \in \mathbb{R}$$

$$M_0 = \Omega^{0000}$$
$$M_1 = 4\Omega^{000i}k_i$$
$$M_2 = 6\Omega^{00ij}k_ik_j$$
$$\Omega^{00\alpha\beta}(\chi) = \Omega^{00\alpha\beta}\left({}^{(p)}\chi\right)$$

- Reality : $b \leq 0 \Rightarrow 4M_0M_2 \leq M_1^2$
- Time reversal : $M_1 = M_3 = 0$

$$\Rightarrow \omega^2 + \frac{1}{2}\frac{M_2}{M_0} = \omega^2 + \frac{3\Omega^{00ij}}{M_0}k_ik_j = \eta^{ab}K_aK_b = 0$$

$$\boxed{\begin{aligned}\eta^{00} &= 1\\ \eta^{ij} &= \frac{3\Omega^{00ij}}{M_0}\end{aligned}}$$

Metric tensor components

1) The metric has a lorentzian signature.

2) Compatible with Hodge duality*.

→ Effective (Gordon) or physical spacetime ?

Effective geometry ?

FW Hehl, YN Obukhov, Found. Phys. **35** (2004)

is intimately related to the minus sign in the reciprocity transformation (52) and the closure relation (54). A plus sign would yield the wrong Euclidean signature. Our approach shows that one can treat the duality operator # as a metricfree predecessor of the Hodge operator $\star$ that appears in the standard Maxwell–Lorentz spacetime relation: # (duality operator) of Eq. (55) $\longrightarrow$ $\star$ (Hodge operator) of Eq. (2).

Summarizing, the conformal part of the metric, that is, the light cone, naturally emerges in our framework from a local and linear spacetime relation. In this sense, the light cone (and the spacetime metric) is an *electromagnetic construct.*

→ **Physical Minkowski spacetime !**

A way out: teleparallelism

R Aldrovandi and JG Pereira. Teleparallel gravity (2013).

	General relativity	Teleparallelism
gravitation is	curvature	torsion

A way out: teleparallelism

R Aldrovandi and JG Pereira. Teleparallel gravity (2013).

	General relativity	Teleparallelism
gravitation is	curvature	torsion
basic field	metric tensor $g_{\mu\nu}$ 10 independent components	tetrad field $e^{a}{}_{\mu}$ 16 independent components

A way out: teleparallelism

R Aldrovandi and JG Pereira. Teleparallel gravity (2013).

	General relativity	Teleparallelism
gravitation is	curvature	torsion
basic field	metric tensor $g_{\mu\nu}$ 10 independent components	tetrad field $e^{a}{}_{\mu} e^{b}{}_{\nu} \eta_{ab} = g_{\mu\nu}$ 16 independent components

A way out: teleparallelism

R Aldrovandi and JG Pereira. Teleparallel gravity (2013).

	General relativity	Teleparallelism
gravitation is	curvature	torsion
basic field	metric tensor $g_{\mu\nu}$ 10 independent components	tetrad field $e^a{}_\mu e^b{}_\nu \eta_{ab} = g_{\mu\nu}$ 16 independent components
metric	lorentzian, dynamic $g_{\mu\nu}$	lorentzian, flat η_{ab}

A way out: teleparallelism

R Aldrovandi and JG Pereira. Teleparallel gravity (2013).

	General relativity	Teleparallelism
gravitation is	curvature	torsion
basic field	metric tensor $g_{\mu\nu}$ 10 independent components	tetrad field $e^a{}_\mu e^b{}_\nu \eta_{ab} = g_{\mu\nu}$ 16 independent components
metric	lorentzian, dynamic $g_{\mu\nu}$	lorentzian, flat η_{ab}
connection	Levi-Civita (torsionless) $\Gamma^\lambda{}_{\mu\rho} = \frac{g^{\lambda\alpha}}{2}\left(\partial_\mu g_{\alpha\rho} + \partial_\rho g_{\alpha\mu} - \partial_\alpha g_{\mu\rho}\right)$	Weitzenböck (curvatureless) $\dot{\Gamma}^\lambda{}_{\mu\rho} = h_a{}^\lambda \partial_\rho e^a{}_\mu = \Gamma^\lambda{}_{\mu\rho} + K^\lambda{}_{\mu\rho}$

A way out: teleparallelism

R Aldrovandi and JG Pereira. Teleparallel gravity (2013).

	General relativity	Teleparallelism
gravitation is	curvature	torsion
basic field	metric tensor $g_{\mu\nu}$ 10 independent components	tetrad field $e^a{}_\mu e^b{}_\nu \eta_{ab} = g_{\mu\nu}$ 16 independent components
metric	lorentzian, dynamic $g_{\mu\nu}$	lorentzian, flat η_{ab}
connection	Levi-Civita (torsionless)	Weitzenböck (curvatureless)
particle trajectory	geodesics (based on WEP) $\frac{du_\mu}{ds} - \Gamma^\lambda{}_{\mu\rho} u_\lambda u^\rho = 0$	force equation $\frac{du_\mu}{ds} - \dot{\Gamma}^\lambda{}_{\mu\rho} u_\lambda u^\rho = T^\lambda{}_{\mu\rho} u_\lambda u^\rho$

A way out: teleparallelism

R Aldrovandi and JG Pereira. Teleparallel gravity (2013).

	General relativity	Teleparallelism
gravitation is	curvature	torsion
basic field	metric tensor $g_{\mu\nu}$ 10 independent components	tetrad field $e^a{}_\mu e^b{}_\nu \eta_{ab} = g_{\mu\nu}$ 16 independent components
metric	lorentzian, dynamic $g_{\mu\nu}$	lorentzian, flat η_{ab}
connection	Levi-Civita (torsionless)	Weitzenböck (curvatureless)
particle trajectory	geodesics (based on WEP) $\frac{du_\mu}{ds} - \Gamma^\lambda{}_{\mu\rho} u_\lambda u^\rho = 0$	force equation $\frac{du_\mu}{ds} - \Gamma^\lambda{}_{\mu\rho} u_\lambda u^\rho = \left(\frac{m_g - m_i}{m_g}\right) \partial_\mu x^a \frac{du_a}{ds}$

A way out: teleparallelism

R Aldrovandi and JG Pereira. Teleparallel gravity (2013).

	General relativity	Teleparallelism
gravitation is	curvature	torsion
basic field	metric tensor $g_{\mu\nu}$ 10 independent components	tetrad field $e^a{}_\mu e^b{}_\nu \eta_{ab} = g_{\mu\nu}$ 16 independent components
metric	lorentzian, dynamic $g_{\mu\nu}$	lorentzian, flat η_{ab}
connection	Levi-Civita (torsionless)	Weitzenböck (curvatureless)
particle trajectory	geodesics (based on WEP)	force equation
		Originates from PRED

Summary

C Pfeifer and D Siemssen. Phys. Rev. D **93** (2016).

- **In 4 dimensions, PRED provides a new look on the fundamental structures of electromagnetism:**

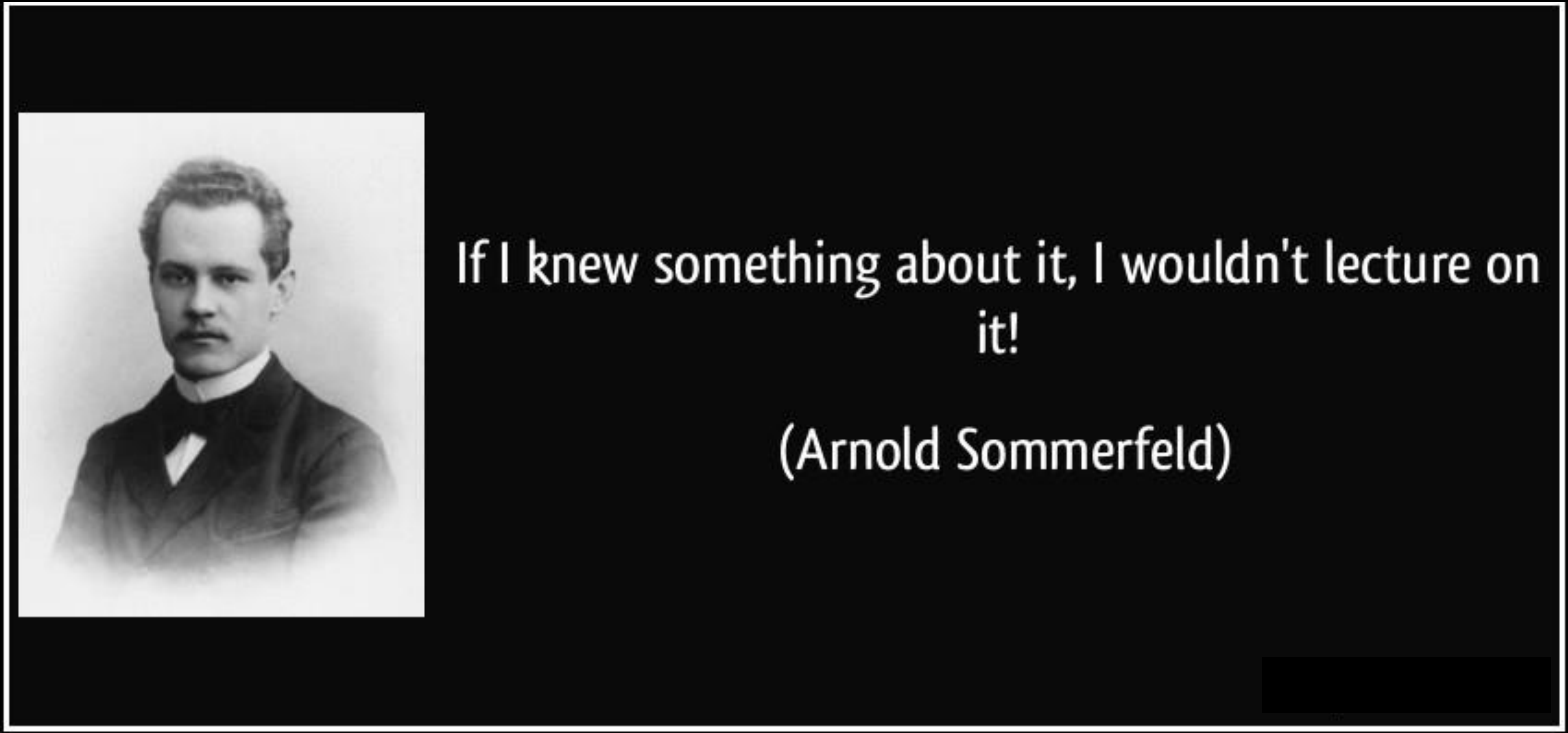
  1. Electromagnetism can be buit from 2 axioms relying only on topology: charge conservation, magnetic flux conservation.

  2. They lead to 4 equations involving 4 fundamental fields: *E*, *D, H* and *B*. These equations are not obtained from averaging processes and they are valid at a microscopic scale.

  3. Maxwell theory is only one element of a larger set of gauge theories that can be built from the same set of axioms = that can legitimately be called electrodynamics.

- **Without vacuum birefringence, the metric of spacetime originates from PRED topological equations.**

- **Such theory was successfully quantized*.**


If I knew something about it, I wouldn't lecture on it!
(Arnold Sommerfeld)
[redacted]

The case of dielectric matter

• A metric was obtained from a general dispersion relation in vacuum. What does it become for a dispersion relation in matter (comoving frame) ?

$$-n^2\frac{\omega^2}{c^2}+K^2=0 \Leftrightarrow -\frac{\omega^2}{c^2}+K^2+(1-n^2)\frac{\omega^2}{c^2}=0 \Leftrightarrow K_\mu\ \eta^{\mu\nu}K_\nu+(1-n^2)\frac{\omega^2}{c^2}=0$$

$$\left.\begin{array}{l} K_\mu=\begin{pmatrix}-\omega/c\\ \mathbf{k}\end{pmatrix} \\ V^\mu=\gamma\begin{pmatrix}1\\ \mathbf{v}/c\end{pmatrix}\Rightarrow V^\mu=\begin{pmatrix}1\\ 0\end{pmatrix}\end{array}\right]\Rightarrow -\frac{\omega}{c}=K_\mu\ V^\mu$$

$$\Rightarrow 0=K_\mu\ \eta^{\mu\nu}K_\nu+(1-n^2)K_\mu\ V^\mu K_\nu\ V^\nu=K_\mu\ g^{\mu\nu}K_\nu$$

W Gordon. Ann. Phys. Leipzig **72** (1923).

Gordon metric tensor

$$g^{\mu\nu} = \eta^{\mu\nu} + (1-n^2)\, V^{\mu}\, V^{\nu}$$

$$g_{\mu\nu} = \eta_{\mu\nu} + \left(1-\frac{1}{n^2}\right) V_{\mu}\, V_{\nu}$$

Fresnel dragging coefficient

- Dielectric dispersion relation = light-cone condition in a Gordon spacetime.
- For vacuum, Fresnel's term is lost and only the lorentzian part remains, as obtained from the premetric approach ⇒ the emerging geometry is only coupled to EM fields but not to other fields : the spacetime is effective but not physical.

Dimensions of space and time

M Tegmark. Class. Quantum Grav. **14** 69-75 (1997)

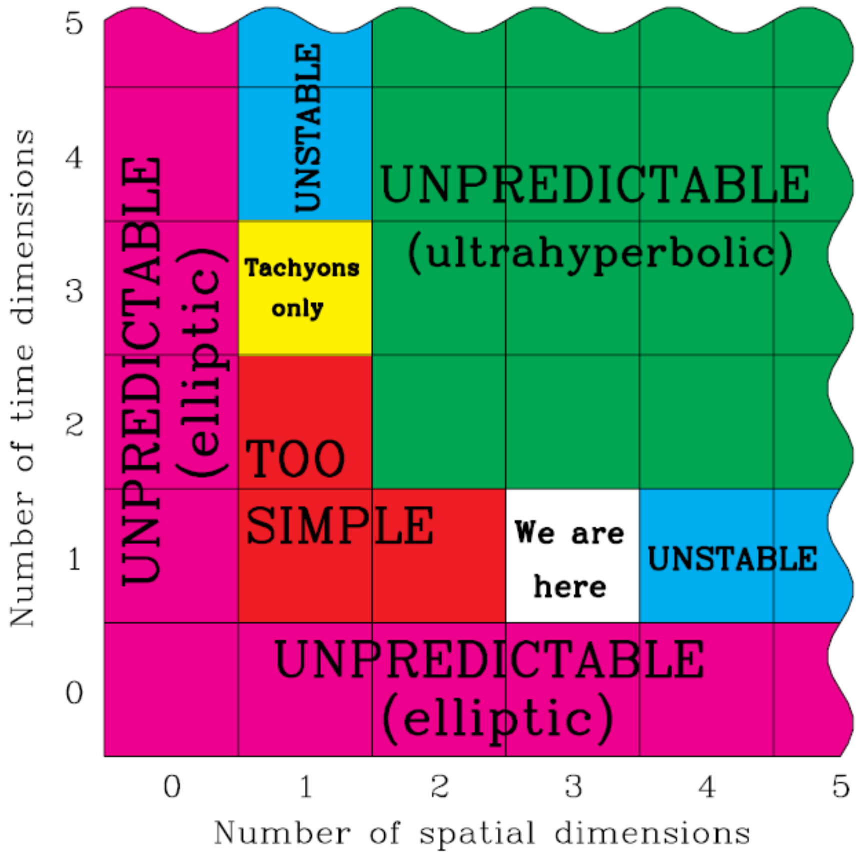


Figure 1. When the partial differential equations of nature are elliptic or ultrahyperbolic, physics has no predictive power for an observer. In the remaining (hyperbolic) cases, $n > 3$ may fail on the stability requirement (atoms are unstable) and $n < 3$ may fail on the complexity requirement (no gravitational attraction, topological problems).

Hafele-Keating experiment

JC Hafele, RE Keating. Science, **177** 4044 (1972)

Around-the-World Atomic Clocks: Predicted Relativistic Time Gains

Abstract. *During October 1971, four cesium beam atomic clocks were flown on regularly scheduled commercial jet flights around the world twice, once eastward and once westward, to test Einstein's theory of relativity with macroscopic clocks. From the actual flight paths of each trip, the theory predicts that the flying clocks, compared with reference clocks at the U.S. Naval Observatory, should have lost 40 ± 23 nanoseconds during the eastward trip, and should have gained 275 ± 21 nanoseconds during the westward trip. The observed time differences are presented in the report that follows this one.*

One of the most enduring scientific debates of this century is the relativistic clock "paradox" (*1*) or problem (*2*), which stemmed originally from an alleged logical inconsistency in predicted differences are compared with our observed time differences in the following report.

A brief elementary review of the theory seems appropriate, particularly surface in the equatorial plane with a ground speed v has a coordinate speed $R\Omega + v$, and hence runs slow with a corresponding time ratio $1 - (R\Omega + v)^2/2c^2$. Therefore, if τ and τ_0 are the respective times recorded by the flying and ground reference clocks during a complete circumnavigation, their time difference, to a first approximation, is given by

$$\tau - \tau_0 = -(2R\Omega v + v^2)\tau_0/2c^2 \qquad (1)$$

Consequently, a circumnavigation in the direction of the earth's rotation (eastward, $v > 0$) should produce a time loss, while one against the earth's rotation (westward, $v < 0$) should produce a time gain for the flying clock if $|v| \sim R\Omega$.

General relativity predicts another effect that (for weak gravitational fields)

↪ **Why is *c* involved in an experiment that is not governed by EM phenomena ?**

Playtime

Stigler's law of eponymy: no scientific discovery is named after its original discoverer.

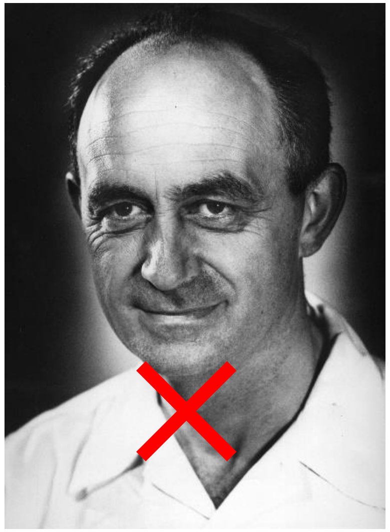
$$\vec{F} = q\,\vec{v} \wedge \vec{B}$$

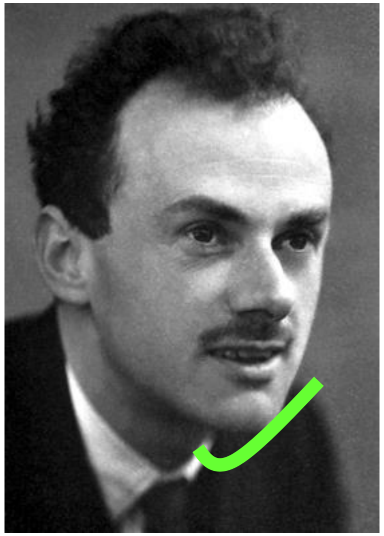
O Heaviside. On the Electromagnetic Effects due to the Motion of Electrification through a Dielectric, Philosophical Magazine (1889).

Playtime

Stigler's law of eponymy: no scientific discovery is named after its original discoverer.

$$T_{i\to f} = \frac{2\pi}{\hbar}\left|\left\langle \beta_f \middle| W \middle| \varphi_i \right\rangle\right|^2 \rho_f$$



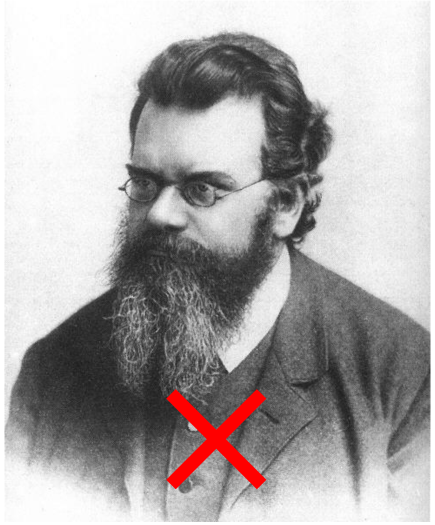


PAM Dirac. The Quantum Theory of Emission and Absorption of Radiation, Proc. Roy. Soc. A (1927).

Playtime

Stigler's law of eponymy: no scientific discovery is named after its original discoverer.

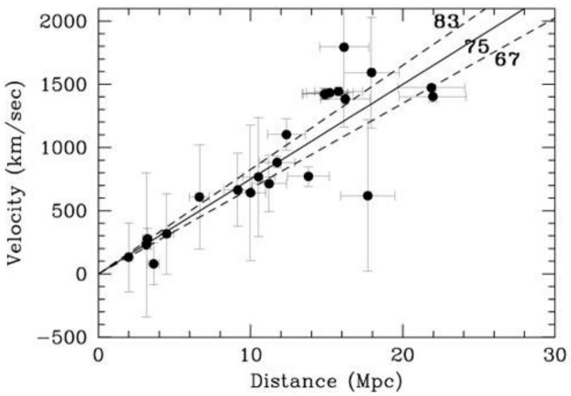
$$k_B = 1{,}38064852(79).10^{-23}\ m^2.kg.s^{-2}.K^{-1}$$



M Planck. On the Law of Distribution of Energy in the Normal Spectrum, Ann. Phys .(1901).

Playtime

Stigler's law of eponymy: no scientific discovery is named after its original discoverer.

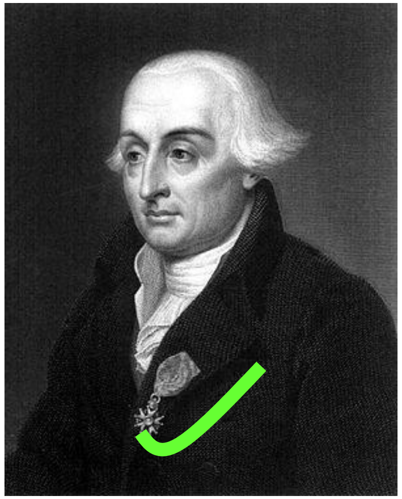


G Lemaître. Annales de la Société Scientifique de Bruxelles 47A, pp. 49–59 (1927).

Playtime

Stigler's law of eponymy: no scientific discovery is named after its original discoverer.

$$\mathbf{H}\,(q, p) = \frac{p^2}{2m} + V(q)$$



J-L Lagrange, Mécanique Analytique, 2nd edition (1811).

290 MÉCANIQUE ANALYTIQUE.

à des centres fixes ou à des corps du même système, et sont fonctions des distances p, q, r, etc., en faisant

$$Pdp + Qdq + Rdr + \text{etc.} = d\Pi,$$

l'équation précédente devient

$$S\left(\frac{dxd^2x + dyd^2y + dzd^2z}{dt^2} + d\Pi\right)\mathrm{m} = 0,$$

dont l'intégrale est

$$S\left(\frac{dx^2 + dy^2 + dz^2}{2dt^2} + \Pi\right)\mathrm{m} = H,$$

dans laquelle H désigne une constante arbitraire, égale à la valeur du premier membre de l'équation dans un instant donné.

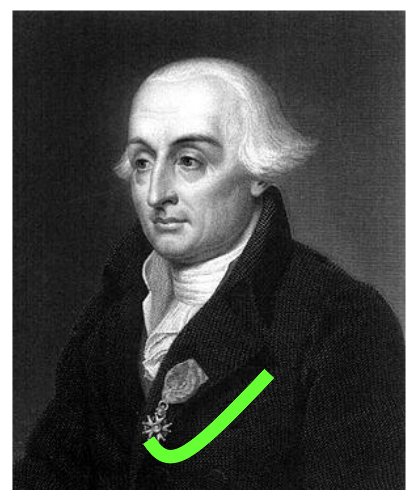
Cette dernière équation renferme le principe connu sous le nom de *Conservation des forces vives*. En effet, $dx^2 + dy^2 + dz^2$ étant le carré de l'espace que le corps parcourt dans l'instant dt, $\frac{dx^2 + dy^2 + dz^2}{dt^2}$ sera le carré de sa vitesse, et $\frac{dx^2 + dy^2 + dz^2}{dt^2}\mathrm{m}$ sa force vive. Donc $S\left(\frac{dx^2 + dy^2 + dz^2}{dt^2}\right)\mathrm{m}$ sera la somme des forces vives de tous les corps, ou la force vive de tout le système ; et on voit par l'équation dont il s'agit, que cette force vive est égale à la quantité $2H - 2S\Pi\mathrm{m}$, laquelle dépend simplement des forces accélératrices qui agissent sur les corps, et nullement de leur liaison mutuelle, de sorte que la force vive du système est à chaque instant la même que les corps auraient acquise si étant animés par les mêmes puissances, ils s'étaient mus librement chacun sur la ligne qu'il a décrite. C'est ce qui a fait donner le nom de *Conservation des forces vives*, à cette propriété du mouvement.

Playtime

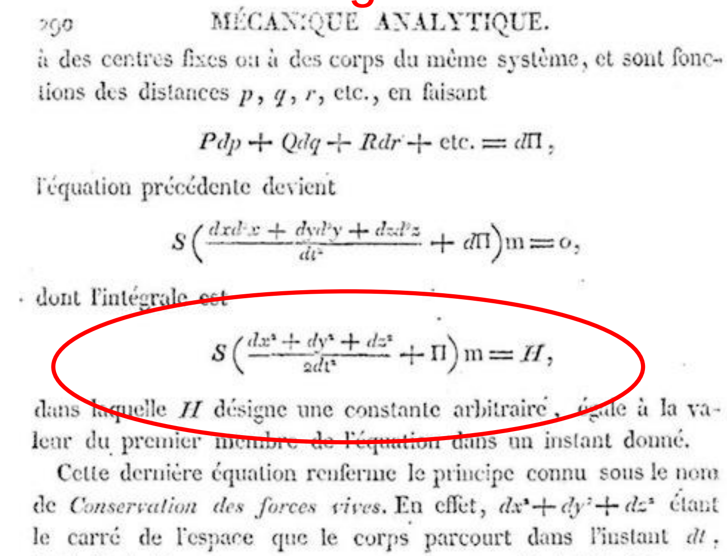
Stigler's law of eponymy: no scientific discovery is named after its original discoverer.

$$\mathbf{H}(q, p) = \frac{p^2}{2m} + V(q)$$

(1805-1865)



J-L Lagrange, Mécanique Analytique, 2nd edition (1811).



290 MÉCANIQUE ANALYTIQUE.

à des centres fixes ou à des corps du même système, et sont fonctions des distances p, q, r, etc., en faisant

$$Pdp + Qdq + Rdr + \text{etc.} = d\Pi,$$

l'équation précédente devient

$$S\left(\frac{dxd^2x + dyd^2y + dzd^2z}{dt^2} + d\Pi\right)\mathrm{m} = 0,$$

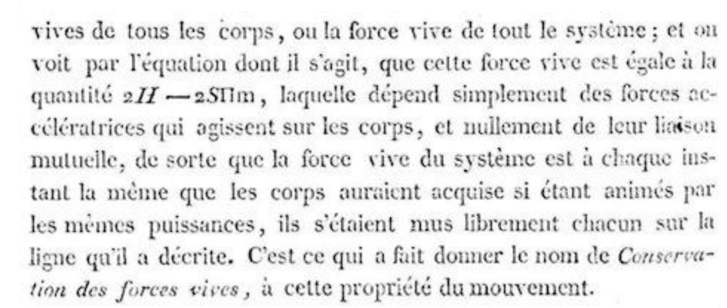
dont l'intégrale est

$$S\left(\frac{dx^2 + dy^2 + dz^2}{2dt^2} + \Pi\right)\mathrm{m} = H,$$

dans laquelle H désigne une constante arbitraire, égale à la valeur du premier membre de l'équation dans un instant donné.

Cette dernière équation renferme le principe connu sous le nom de *Conservation des forces vives*. En effet, $dx^2 + dy^2 + dz^2$ étant le carré de l'espace que le corps parcourt dans l'instant dt,

Why using the letter "*H*" ?

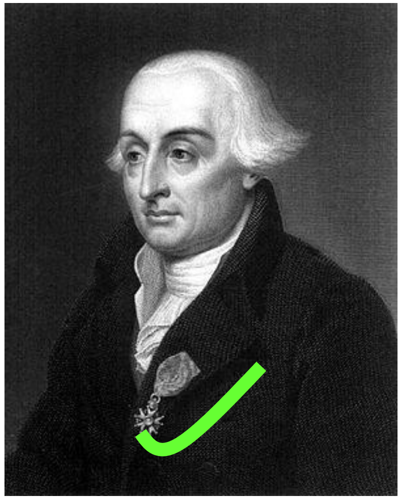


vives de tous les corps, ou la force vive de tout le système; et on voit par l'équation dont il s'agit, que cette force vive est égale à la quantité $2H - 2S\Pi\mathrm{m}$, laquelle dépend simplement des forces accélératrices qui agissent sur les corps, et nullement de leur liaison mutuelle, de sorte que la force vive du système est à chaque instant la même que les corps auraient acquise si étant animés par les mêmes puissances, ils s'étaient mus librement chacun sur la ligne qu'il a décrite. C'est ce qui a fait donner le nom de *Conservation des forces vives*, à cette propriété du mouvement.

Playtime

Stigler's law of eponymy: no scientific discovery is named after its original discoverer.

$$\mathbf{H}(q, p) = \frac{p^2}{2m} + V(q)$$



J-L Lagrange, Mécanique Analytique,
2nd edition (1811).